Free Vibration Analysis of Thick Isotropic Plate by Using 5th Order Shear Deformation Theory

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Abstract: In the present study, a fifth order shear and normal deformation theory is presented for free vibration analysis of simply supported thick isotropic square plates. Governing equations and boundary conditions are obtained using the virtual work principle. The bending, thickness shear, and thickness stretch frequencies are obtained using the Navier solution technique for simply supported thick isotropic plates. The eigenvalue problem in this study is solved using the MATLAB program. Numerical results for free vibration analysis include the effect of side to thickness ratio for simply supported thick isotropic square plates. The bending and thickness shear mode frequencies obtained using the present theory are in excellent agreement with those of exact solution and other refined shear deformation theories. Further, for the first time, the thickness stretch mode frequencies are also presented using this theory.

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1. Introduction

Plates are the most widely used structural element in the field of engineering. Behaviour of the thick plates either, isotropic or anisotropic, subjected to static and dynamic loading requires the use of refined shear deformation theories, as the classical plate theory neglects the effect of transverse shear deformation generated in the thick plates. The classical plate theory based on the Navier-Kirchhoff hypothesis [1, 2] under predicts deflections and over predicts natural frequencies. Further, the primary concern of the researchers is to develop an equivalent one-dimensional (1-D) and two-dimensional (2-D) theories for the three-dimensional (3-D) plate problems.

Timoshenko and Woinowsky-Krieger [3] presented flexural and buckling solutions for different types of thin plates subjected to various static loading and boundary

conditions. Jemielita [4] introduced a brief review of the development in the theory of plates formulation from 1789-1988. Refined shear deformation theories presented by Reissner [5], Levy [6], Mindlin [7], Hencky [8] and Kromm [9] are the refinement of the classical plate theory (CPT) considering the effect of transverse shear deformation. Shear deformation theories depending on the assumption of primary variables can be classified as stress-based and displacement-based theories. Reissner's theory is stress based and Hencky, Mindlin theories are displacement-based theories.

The first refined higher order theory was proposed by Levy in 1877 [6]. A century later, Lo et al. [11, 10] established a compatible higher order theory for laminated plates. The proposed theory considered the effects of transverse normal strain, transverse shear deformation, in-plane and transverse displacements with respect to the thickness

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coordinate. Many higher order refined shear deformation theories have been developed in the last three decades. Various authors have presented critical reviews [12, 13] on the refined shear deformation theories. A review of some of the prominent theories is presented below.

Srinivas et al. [14] presented a three-dimensional linear, small deformation theory for the free vibration of simply-supported, homogeneous, isotropic, thick rectangular plates. An improved third order shear deformation theory is formulated by Shi [15]. Akhavan et al. [16] adopted the differential transformation method to establish a semi-analytical solution for free vibration and modal stress studies of circular plates resting on two-parameter elastic foundations. Sayyad and Ghugal [17,18] provided exponential shear deformation theory (ESDT) and (TSDT) trigonometric shear deformation theory for bending and free vibration analysis of various thick plates. Reddy [19, 20] proposed a third order shear deformation theory for the static and dynamic analyses of isotropic and composite plates. An exact analytical solution for free vibration analysis of thick laminated annular/circular plates is presented by Serajian and Asadi [21]. Thai and Kim [22] presented a two variable refined plate theory for free vibration analysis of cross-ply and angle-pylaminated plates. Amale et al. [23] provided a new hyperbolic shear deformation theory that considers parabolic transverse shear strains across the thickness direction for free vibration analysis of plates. The exact elasticity solution for bidirectional bending of plates is provided by Pagano [24].

In the literature review presented above, it's seen that the third order shear deformation theories have been widely used for the analysis of all types of plates to predict their true behaviour when subjected to various kinds of loads. In the analysis of both thick and thin plates subjected to transverse thermal loading, plate theories that have not considered normal strain lead to inaccurate results because of the isotropic nature of the thermal loading. To overcome, it is required that at least transverse displacement function contains a parabolic expansion term. Keeping this in view, Carrera [25], Rohwer et al. [26] suggested the plate theories with a fifth or higher order expansion of in-plane displacements and fourth order expansion of transverse displacement in thickness coordinate of plate to include transverse normal strain effects.

Considering the above stated fact, a modified fifth order shear deformation theory [27] is presented for the free

vibration analysis of thick isotropic square plates. The higher frequencies corresponding to the thickness stretch mode cannot be obtained using the classical plate and existing higher order shear deformation theories with zero transverse normal strain. The solution of the eigenvalue problem yields the frequencies corresponding to bending, thickness shear and thickness stretch modes.

2. Mathematical Formulation for 5th order shear deformation theory

Consider an undeformed plate of length a , width b , and thickness h composed of isotropic, homogenous material. The Cartesian coordinate system is such that the middle plane of the plate coincides with the xy plane, and the z axis is normal to the middle plane. The plate is bounded by the coordinate planes $x=0, a$ and $y=0, b$. The reference surface is the plate's middle surface defined by $z=0$ and z represents the thickness coordinate measured from the undeformed middle surface. Thus, the plate occupies in $O-x-y-z$ right-handed rectangular Cartesian Coordinate system, a region: $0 \leq x \leq a; 0 \leq y \leq b; -h/2 \leq z \leq h/2$ (1)

The displacement function of the fifth order shear deformation theory is given as,

$$\begin{aligned} U(x, y, z, t) &= u_0 - z \frac{\partial w}{\partial x} + f_1(z) \phi_x(x, y, t) + f_2(z) \psi_x(x, y, t) \\ V(x, y, z, t) &= v_0 - z \frac{\partial w}{\partial y} + f_1(z) \phi_y(x, y, t) + f_2(z) \psi_y(x, y, t) \\ W(x, y, z, t) &= w(x, y, t) + g_1(z) \phi_z(x, y, t) + g_2(z) \psi_z(x, y, t) \end{aligned} \quad (2)$$

$$\text{where } f_1(z) = z \left(1 - \frac{4}{3} \frac{z^2}{h^2} \right), \quad f_2(z) = z \left(1 - \frac{16}{5} \frac{z^4}{h^4} \right)$$

$$\text{and } g_1(z) = \left(1 - 4 \frac{z^2}{h^2} \right), \quad g_2(z) = \left(1 - 16 \frac{z^4}{h^4} \right)$$

where U, V, W , are the displacements in the x, y, z , direction, respectively. $\phi_x, \psi_x, \phi_y, \psi_y, \phi_z, \psi_z$ are the shear rotations due to bending in x, y , and z directions. u_0, v_0 and w are the middle and transverse displacements in x, y and z directions, respectively. The functions $f(z)$ and $g(z)$ are the transverse shear and normal strain functions, respectively. Fig. 1 (a) shows the variation of transverse shear function $f_1(z)$ and $f_2(z)$ with thickness. Similarly, Fig. 1 (b) shows the variation of the first derivative of the transverse shear function, i.e., $g_1(z)$ and $g_2(z)$ with thickness. It is observed from the figures that transverse shear functions and their first derivatives are symmetrical with respect to the mid plane of the plate ($z=0$).

Fig. 1 (a) Transverse shear functions $f_1(z)$ and $f_2(z)$ v/s thickness and **(b)** First derivative of transverse shear functions, i.e., $g_1(z)$ and $g_2(z)$ v/s thickness.

The normal strains $\varepsilon_x, \varepsilon_y, \varepsilon_z$ and shear strains $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ are obtained within the framework of linear theory of elasticity.

$$\begin{aligned}
 \varepsilon_x &= \frac{\partial U}{\partial x} = \frac{\partial u_0}{\partial x} - z \frac{\partial^2 w}{\partial x^2} + f_1(z) \frac{\partial^2 \phi_x}{\partial x^2} + f_2(z) \frac{\partial^2 \psi_x}{\partial x^2} \\
 \varepsilon_y &= \frac{\partial V}{\partial y} = \frac{\partial v_0}{\partial y} - z \frac{\partial^2 w}{\partial y^2} + f_1(z) \frac{\partial^2 \phi_y}{\partial y^2} + f_2(z) \frac{\partial^2 \psi_y}{\partial y^2} \\
 \varepsilon_z &= \frac{\partial W}{\partial z} = -8 \frac{z}{h^2} \phi_z - 64 \frac{z^3}{h^4} \psi_z \\
 \gamma_{xy} &= \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w}{\partial x \partial y} + f_1(z) \frac{\partial \phi_x}{\partial y} \\
 &\quad + f_1(z) \frac{\partial \phi_y}{\partial x} + f_2(z) \frac{\partial \psi_x}{\partial y} + f_2(z) \frac{\partial \psi_y}{\partial x} \\
 \gamma_{zx} &= \frac{\partial U}{\partial z} + \frac{\partial W}{\partial x} = f_1'(z) \phi_x + f_2'(z) \psi_x + g_1(z) \frac{\partial \phi_z}{\partial x} + g_2(z) \frac{\partial \psi_z}{\partial x} \\
 \gamma_{yz} &= \frac{\partial V}{\partial z} + \frac{\partial W}{\partial y} = f_1'(z) \phi_y + f_2'(z) \psi_y + g_1(z) \frac{\partial \phi_z}{\partial y} + g_2(z) \frac{\partial \psi_z}{\partial y}
 \end{aligned} \quad (3)$$

The stress strain relationship for the linear isotropic elastic plate can be written as:

$$\begin{aligned}
 \sigma_x &= \lambda (\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G \varepsilon_x \\
 \sigma_y &= \lambda (\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G \varepsilon_y \\
 \sigma_z &= \lambda (\varepsilon_x + \varepsilon_y + \varepsilon_z) + 2G \varepsilon_z \\
 \tau_{xy} &= G \gamma_{xy}, \tau_{yz} = G \gamma_{yz}, \tau_{zx} = G \gamma_{zx}
 \end{aligned} \quad (4)$$

Where λ and G are the Lamé's constants as given below

$$\lambda = E \mu / [(1 + \mu)(1 - 2\mu)] \text{ and } G = E / [2(1 + \mu)]$$

in which E is the Young's modulus, G is the shear modulus and μ is the Poisson's ratio.

2.1. Governing equations and boundary condition

Using the principle of virtual work, strain displacement relationship and constitutive equations of isotropic material, the governing differential equations and boundary conditions can be obtained. The principle of virtual work applied to the considered plate leads to,

$$\begin{aligned}
 &\int_{z=-h/2}^{z=h/2} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ \sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \sigma_z \delta \varepsilon_z \right. \\
 &\quad \left. + \tau_{xy} \delta \gamma_{xy} + \tau_{yz} \delta \gamma_{yz} + \tau_{zx} \delta \gamma_{zx} \right\} dx dy dz \\
 &+ \rho \int_{z=-h/2}^{z=h/2} \int_{y=0}^{y=b} \int_{x=0}^{x=a} \left\{ \frac{\partial^2 u}{\partial t^2} \delta u + \frac{\partial^2 v}{\partial t^2} \delta v + \frac{\partial^2 w}{\partial t^2} \delta w \right\} dx dy dz \\
 &- \int_{y=0}^{y=b} \int_{x=0}^{x=a} q(x, y) \delta w dx dy = 0
 \end{aligned} \quad (5)$$

where symbol δ denotes the variation of displacement variables. Employing Green's theorem in Eq.(5) successively, we obtained the coupled Euler-Lagrange equations, which are the governing equations and the associated boundary conditions of the plate using fundamental lemma of calculus of variations as follows:

$$\begin{aligned}
 \delta w : & L_{11}w + L_{12}\phi_x + L_{13}\psi_x + L_{14}\phi_y + L_{15}\psi_y + L_{16}\phi_z + L_{17}\psi_z = q \\
 \delta \phi_x : & L_{21}w + L_{22}\phi_x + L_{23}\psi_x + L_{24}\phi_y + L_{25}\psi_y + L_{26}\phi_z + L_{27}\psi_z = 0 \\
 \delta \psi_x : & L_{31}w + L_{32}\phi_x + L_{33}\psi_x + L_{34}\phi_y + L_{35}\psi_y + L_{36}\phi_z + L_{37}\psi_z = 0 \\
 \delta \phi_y : & L_{41}w + L_{42}\phi_x + L_{43}\psi_x + L_{44}\phi_y + L_{45}\psi_y + L_{46}\phi_z + L_{47}\psi_z = 0 \\
 \delta \psi_y : & L_{51}w + L_{52}\phi_x + L_{53}\psi_x + L_{54}\phi_y + L_{55}\psi_y + L_{56}\phi_z + L_{57}\psi_z = 0 \\
 \delta \phi_z : & L_{61}w + L_{62}\phi_x + L_{63}\psi_x + L_{64}\phi_y + L_{65}\psi_y + L_{66}\phi_z + L_{67}\psi_z = 0 \\
 \delta \psi_z : & L_{71}w + L_{72}\phi_x + L_{73}\psi_x + L_{74}\phi_y + L_{75}\psi_y + L_{76}\phi_z + L_{77}\psi_z = 0
 \end{aligned} \quad (6)$$

The functional operators, L_{ij} , ($i, j = 1, 2, \dots, 7$) etc. are summarized in Appendix A.

The boundary conditions along the edges $x=0$ and $x=a$ obtained are of the following form:

$$\begin{aligned}
& L_{11}^0 w + L_{12}^0 \phi_x + L_{13}^0 \psi_x + L_{14}^0 \phi_y + L_{15}^0 \psi_y + L_{16}^0 \phi_z + L_{17}^0 \psi_z = 0 \\
& \text{or } w \text{ is prescribed.} \\
& L_{21}^0 w + L_{22}^0 \phi_x + L_{23}^0 \psi_x + L_{24}^0 \phi_y + L_{25}^0 \psi_y + L_{26}^0 \phi_z + L_{27}^0 \psi_z = 0 \\
& \text{or } \frac{\partial w}{\partial x} \text{ is prescribed.} \\
& L_{31}^0 w + L_{32}^0 \phi_x + L_{33}^0 \psi_x + L_{34}^0 \phi_y + L_{35}^0 \psi_y + L_{36}^0 \phi_z + L_{37}^0 \psi_z = 0 \\
& \text{or } \phi_x \text{ is prescribed.} \\
& L_{41}^0 w + L_{42}^0 \phi_x + L_{43}^0 \psi_x + L_{44}^0 \phi_y + L_{45}^0 \psi_y + L_{46}^0 \phi_z + L_{47}^0 \psi_z = 0 \\
& \text{or } \psi_x \text{ is prescribed.} \\
& L_{51}^0 w + L_{52}^0 \phi_x + L_{53}^0 \psi_x + L_{54}^0 \phi_y + L_{55}^0 \psi_y + L_{56}^0 \phi_z + L_{57}^0 \psi_z = 0 \\
& \text{or } \phi_y \text{ is prescribed.} \\
& L_{61}^0 w + L_{62}^0 \phi_x + L_{63}^0 \psi_x + L_{64}^0 \phi_y + L_{65}^0 \psi_y + L_{66}^0 \phi_z + L_{67}^0 \psi_z = 0 \\
& \text{or } \psi_y \text{ is prescribed.} \\
& L_{71}^0 w + L_{72}^0 \phi_x + L_{73}^0 \psi_x + L_{74}^0 \phi_y + L_{75}^0 \psi_y + L_{76}^0 \phi_z + L_{77}^0 \psi_z = 0 \\
& \text{or } \phi_z \text{ is prescribed.} \\
& L_{81}^0 w + L_{82}^0 \phi_x + L_{83}^0 \psi_x + L_{84}^0 \phi_y + L_{85}^0 \psi_y + L_{86}^0 \phi_z + L_{87}^0 \psi_z = 0 \\
& \text{or } \psi_z \text{ is prescribed.}
\end{aligned}
\tag{7}$$

The functional operators L_{ij}^0 etc. are summarized in Appendix A.

The boundary conditions at $y = 0$ and $y = b$ obtained is of the following form:

$$\begin{aligned}
& L_{S11} w + L_{S12} \phi_x + L_{S13} \psi_x + L_{S14} \phi_y + L_{S15} \psi_y + L_{S16} \phi_z + L_{S17} \psi_z = 0 \\
& \text{or } w \text{ is prescribed.} \\
& L_{S21} w + L_{S22} \phi_x + L_{S23} \psi_x + L_{S24} \phi_y + L_{S25} \psi_y + L_{S26} \phi_z + L_{S27} \psi_z = 0 \\
& \text{or } \frac{\partial w}{\partial x} \text{ is prescribed.} \\
& L_{S31} w + L_{S32} \phi_x + L_{S33} \psi_x + L_{S34} \phi_y + L_{S35} \psi_y + L_{S36} \phi_z + L_{S37} \psi_z = 0 \\
& \text{or } \phi_x \text{ is prescribed.} \\
& L_{S41} w + L_{S42} \phi_x + L_{S43} \psi_x + L_{S44} \phi_y + L_{S45} \psi_y + L_{S46} \phi_z + L_{S47} \psi_z = 0 \\
& \text{or } \psi_x \text{ is prescribed.} \\
& L_{S51} w + L_{S52} \phi_x + L_{S53} \psi_x + L_{S54} \phi_y + L_{S55} \psi_y + L_{S56} \phi_z + L_{S57} \psi_z = 0 \\
& \text{or } \phi_y \text{ is prescribed.} \\
& L_{S61} w + L_{S62} \phi_x + L_{S63} \psi_x + L_{S64} \phi_y + L_{S65} \psi_y + L_{S66} \phi_z + L_{S67} \psi_z = 0 \\
& \text{or } \psi_y \text{ is prescribed.} \\
& L_{S71} w + L_{S72} \phi_x + L_{S73} \psi_x + L_{S74} \phi_y + L_{S75} \psi_y + L_{S76} \phi_z + L_{S77} \psi_z = 0 \\
& \text{or } \phi_z \text{ is prescribed.} \\
& L_{S81} w + L_{S82} \phi_x + L_{S83} \psi_x + L_{S84} \phi_y + L_{S85} \psi_y + L_{S86} \phi_z + L_{S87} \psi_z = 0 \\
& \text{or } \psi_z \text{ is prescribed.}
\end{aligned}
\tag{8}$$

The boundary condition at corners $(x=0, y=0)$, $(x=a, y=0)$, $(x=0, y=b)$ and $(x=a, y=b)$ is obtained in terms of stress resultant is of the following form:

$$L_{P81} w + L_{P82} \phi_x + L_{P83} \psi_x + L_{P84} \phi_y + L_{P85} \psi_y + L_{P86} \phi_z + L_{P87} \psi_z = 0 \tag{9}$$

or w is prescribed.

The functional operators L_{Sij}, L_{Pij} etc. are summarized in Appendix A.

The plate stiffness constants and inertia terms appeared in the governing equations and boundary conditions are expressed as follows:

$$\begin{aligned}
& \left\{ \begin{matrix} D_1, F_1, L_1, \\ MA_1, NA_1, \\ S_1, P_1, R_1, \\ T_1, O_1 \end{matrix} \right\} = (\lambda + 2G) \int_{-h/2}^{h/2} \left\{ \begin{matrix} z^2, z^4, z f_1(z), z f_2(z), f_1(z)^2, \\ f_1(z) f_2(z), f_2(z)^2, z^3 f_1(z), \\ z^3 f_2(z), z^6 \end{matrix} \right\} dz \\
& \left\{ \begin{matrix} D_2, F_2, L_2, \\ MA_2, NA_2, \\ S_2, P_2, R_2, T_2 \end{matrix} \right\} = (\lambda) \int_{-h/2}^{h/2} \left\{ \begin{matrix} z^2, z^4, z f_1(z), z f_2(z), f_1(z)^2, \\ f_1(z) f_2(z), f_2(z)^2, z^3 f_1(z), \\ z^3 f_2(z) \end{matrix} \right\} dz \\
& \left\{ \begin{matrix} C_3, D_3, F_3, G_3, \\ H_3, L_3, MA_3, NA_3, \\ S_3, P_3, R_3, T_3, O_3 \end{matrix} \right\} = (G) \int_{-h/2}^{h/2} \left\{ \begin{matrix} g_1(z)^2, z^2, z^4, g_2(z)^2, \\ g_1(z) g_2(z), z f_1(z), z f_2(z), \\ f_1(z)^2, f_1(z) f_2(z), f_2(z)^2, \\ z^3 f_1(z), z^3 f_2(z), z^6 \end{matrix} \right\} dz
\end{aligned}
\tag{10}$$

3. Illustrative example

Free vibration analysis of simply supported isotropic square plate for different side to thickness ratio is carried out by discarding the transverse mechanical load acting on the plate. The material properties for thick isotropic plate for free vibration analysis are as follows:

$$\begin{aligned}
E &= 210 \text{ GPa,} \\
G &= \frac{E}{2(1+\mu)}, \\
\rho &= 7800 \text{ kg/m}^3, \mu = 0.3
\end{aligned}
\tag{11}$$

3.1 Navier's Solution for free vibration of plate

Simply supported isotropic square plate is considered for the numerical study. The governing differential equations (Eqs. 6) in terms of displacement variables are used by setting the external transverse load q equal to zero. The associated boundary condition for free vibration of plates under consideration can be obtained directly from Eqs. (7) and (8). The following boundary conditions are imposed at simply supported edges:

$$\begin{aligned}
w = \phi_x = \psi_x = \phi_z = \psi_z &= 0 \text{ at } x=0, x=a \\
w = \phi_y = \psi_y = \phi_z = \psi_z &= 0 \text{ at } y=0, y=b
\end{aligned}
\tag{12}$$

The unknown variables form $w(x, y, t)$, $\phi_x(x, y, t)$, $\psi_x(x, y, t)$, $\phi_y(x, y, t)$, $\psi_y(x, y, t)$, $\phi_z(x, y, t)$, $\psi_z(x, y, t)$ are represented in the following trigonometric form satisfying the governing equations and the boundary conditions (time dependent).

$$\begin{aligned}
(w) &= \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} (w_{mn}) \sin \alpha x \sin \beta y \sin \omega_{mn} t \\
(\phi_x, \psi_x) &= \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} (\phi_{xmn}, \psi_{xmn}) \cos \alpha x \sin \beta y \sin \omega_{mn} t \\
(\phi_y, \psi_y) &= \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} (\phi_{ymn}, \psi_{ymn}) \sin \alpha x \cos \beta y \sin \omega_{mn} t \\
(\phi_z, \psi_z) &= \sum_{m=1,3,5,\dots}^{\infty} \sum_{n=1,3,5,\dots}^{\infty} (\phi_{zmn}, \psi_{zmn}) \sin \alpha x \sin \beta y \sin \omega_{mn} t
\end{aligned} \quad (13)$$

where $\omega_{mn}, \phi_{xmn}, \psi_{xmn}, \phi_{ymn}, \psi_{ymn}, \phi_{zmn}, \psi_{zmn}$ are the unknown coefficients of the respective Fourier expansions. Substitution of Eqs. (13) into governing equations, yields the following eigenvalue equation,

$$([K_{ij}] - \omega_{mn}^2 [M_{ij}]) \{\Delta\} = 0 \quad (14)$$

where the vector of amplitude of motion is given by

$$\{\Delta\} = \{w_{mn}, \phi_{xmn}, \psi_{xmn}, \phi_{ymn}, \psi_{ymn}, \phi_{zmn}, \psi_{zmn}\}^T.$$

The elements of stiffness matrix $[K]$ and elements of mass matrix $[M]$, are summarized in Appendix A.

4. Numerical results

The results for natural frequencies ω_{mn} are presented in the following non-dimensional form, $\bar{\omega}_{mn} = \omega_{mn} h \sqrt{\rho/G}$

where ω_{mn} is natural frequency for (m, n) mode, ' h ' is thickness of plate, ' ρ ' is density of material.

Table 1 shows the comparison of non-dimensional bending, thickness shear, and thickness stretch mode frequencies for simply supported thick isotropic square plates by various shear deformation theories. Note that the non-dimensional frequency corresponding to bending mode, thickness-shear modes and thickness stretch mode are denoted by $\bar{\omega}_w, \bar{\omega}_{\phi_x}, \bar{\omega}_{\phi_y}, \bar{\omega}_{\psi_x}, \bar{\omega}_{\psi_y}, \bar{\omega}_{\phi_z}$, respectively.

Table 1 Comparison of non-dimensional flexural, thickness shear and thickness stretch mode frequencies of a simply supported square thick isotropic plate for aspect ratio (side to thickness ratio) $a/h = 10$.

b/a	$\bar{\omega}$	(m, n)	Present [5 th OSDT]	Exact [12]	ESDT [16]	Reddy [18]	Mindlin [7]	Ghugal & Sayyad [17]	CPT [1]
1.0	$\bar{\omega}_w$	(1,1)	0.09315	0.0932	0.0933	0.0931	0.0930	0.0931	0.0955
		(1,2)	0.18088	0.2226	0.2231	0.2219	0.2219	0.2223	0.2360
		(1,3)	0.16006	0.4171	0.4184	0.4150	0.4149	0.4163	0.4629
		(2,2)	0.34207	0.3421	0.3431	0.3406	0.3406	0.3415	0.3732
		(2,3)	0.37138	0.5239	0.5258	0.5208	0.5206	0.5228	0.5951
		(2,4)	0.65542	0.7511	0.7542	0.7453	0.7446	0.7499	0.8926
		(3,3)	0.68893	0.6889	0.6917	0.6839	0.6834	0.6874	0.8090
		(4,4)	1.08880	1.0889	1.0945	1.0758	1.0764	1.0872	1.3716
1.0	$\bar{\omega}_{\phi_x}$	(1,1)	3.1740	3.1729	3.1626	3.1749	3.1730	3.1729	--
		(1,2)	3.2195	3.2192	3.2091	3.2212	3.2193	3.2191	
		(1,3)	3.2951	3.2949	3.2851	3.2969	3.2951	3.2949	
		(2,2)	3.2658	3.2648	3.2549	3.2668	3.2650	3.2648	
		(2,3)	3.3401	3.3396	3.3299	3.3415	3.3397	3.3396	
		(2,4)	3.4418	3.4414	3.4320	3.4433	3.4416	3.4414	
		(3,3)	3.4135	3.4126	3.4031	3.4145	3.4128	3.4126	
		(4,4)	3.6102	3.6094	3.6004	3.6112	3.6096	3.6094	
1.0	$\bar{\omega}_{\psi_x}$	(1,1)	3.1740	3.1729	3.1626	3.1749	3.1730	3.1729	--
		(1,2)	3.2195	3.2192	3.2091	3.2212	3.2193	3.2191	
		(1,3)	3.2951	3.2949	3.2851	3.2969	3.2951	3.2949	
		(2,2)	3.2658	3.2648	3.2549	3.2668	3.2650	3.2648	
		(2,3)	3.3401	3.3396	3.3299	3.3415	3.3397	3.3396	
		(2,4)	3.4418	3.4414	3.4320	3.4433	3.4416	3.4414	
		(3,3)	3.4135	3.4126	3.4031	3.4145	3.4128	3.4126	
		(4,4)	3.6102	3.6094	3.6004	3.6112	3.6096	3.6094	

b/a	(m, n)	Present [5 th OSDT]		
		$\bar{\omega}_{\phi_y}$	$\bar{\omega}_{\psi_y}$	$\bar{\omega}_{\phi_z}$
1.0	(1,1)	9.6107	10.706	30.085
	(1,2)	9.6422	10.669	30.107
	(1,3)	9.6658	10.647	30.118
	(2,2)	9.5799	10.791	30.109
	(2,3)	9.6813	10.676	30.116
	(2,4)	9.6232	10.695	30.123
	(3,3)	9.5462	10.931	30.123
	(4,4)	9.5225	11.115	30.127

It is observed that the predicted bending frequencies for all modes of vibration by the present theory (5th OSDT) are in excellent agreement with those of the exact analytical results. The fundamental mode bending frequencies predicted by 5th OSDT and exact analysis are identical. Further, the trigonometric shear deformation theory (TSDT) overestimates the bending frequency for fundamental mode, and higher shear order deformation theory (HSDT) underestimates the bending frequencies for higher modes. First order shear deformation theory (FSDT) yields lower values of bending frequency for all modes of vibration than those of other higher order and exact results.

In contrast, CPT yields higher values for this frequency. The present theory shows excellent results for thickness shear mode frequency in x direction, i.e., $\bar{\omega}_{\phi_x}, \bar{\omega}_{\psi_x}$ for higher modes. The non-dimensional frequencies corresponding to thickness-shear modes in y direction, i.e., $\bar{\omega}_{\phi_y}, \bar{\omega}_{\psi_y}$ are obtained for the first time. The non-dimensional frequencies corresponding to thickness-stretch mode in z direction, i.e. $\bar{\omega}_{\phi_z}$ are also presented and show values higher than those of other mode frequencies.

Table 2 and Fig.2 show the variation of different frequencies with side to thickness ratio for the first mode (fundamental frequencies) using the present theory. It is observed that after a specific value of side to thickness ratio, all the non-dimensional frequencies attain nearly a constant value since the effect of transverse shear and normal deformations diminishes.

Table 2 Variation of non-dimensional flexural, thickness shear mode and thickness stretch mode frequencies of a simply supported square thick isotropic plate for mode (m=1, n=1) with side to thickness ratio.

(m, n)	$\bar{\omega}$	a/h				
		4	10	20	50	100
(1,1)	$\bar{\omega}_w$	0.5065	0.09315	0.02387	0.003847	0.00096
	$\bar{\omega}_{\phi_x}$	3.333	3.174	3.1497	3.143	3.1432
	$\bar{\omega}_{\psi_x}$	3.7079	3.2467	3.1699	3.1475	3.1432
	$\bar{\omega}_{\phi_y}$	7.3421	9.6107	9.6224	9.6225	9.6226
	$\bar{\omega}_{\psi_y}$	9.8213	10.706	11.501	11.749	11.785
	$\bar{\omega}_{\phi_z}$	23.832	30.085	34.190	36.064	36.374

Fig. 2 Variation of various frequencies with respect to side to thickness (a/h) according to 5th OSDT

5. Conclusions

A fifth order shear and normal deformation theory is utilized for the free vibration analysis of simply supported thick isotropic square plate. The boundary value problem of the plate is obtained by principle of virtual work. The presented theory is novel because it does not require any shear correction factor as the shear stress free boundary conditions are satisfied at the upper and lower surfaces of the plate. Based on the observations of the numerical study, the following conclusions are drawn:

1. Non-dimensional bending and thickness shear modes frequencies obtained for free vibration analysis using the present theory for all modes of vibration agree well with the exact frequencies for the simply supported square plate.
2. Non-dimensional thickness shear mode frequencies in y direction and thickness stretchmode frequencies are obtained for the first time.
3. There exists an optimum value of side to thickness ratio. An increase in side to thickness ratio after that optimum value does not cause much change in the values of non-dimensional frequencies due to the vanishing effect of shear and normal deformations.

Disclosures

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Appendix A

The elements of stiffness matrix [M] are given below:

$$\begin{aligned}
 M_{11} &= [I_1 + I_5(\alpha^2 + \beta^2)]; M_{12} = [-I_6\alpha]; \\
 M_{13} &= [-I_7\alpha]; M_{14} = [-I_6\beta]; M_{15} = [-I_7\beta]; \\
 M_{16} &= [I_{16}]; M_{17} = [I_{17}]; M_{21} = [-I_6\alpha]; \\
 M_{22} &= [I_9]; M_{23} = [I_{10}]; M_{24} = 0; M_{25} = 0; \\
 M_{26} &= 0; M_{27} = 0; M_{31} = [-I_7\alpha]; M_{32} = [I_{10}]; \\
 M_{33} &= [I_{15}]; M_{34} = 0; M_{35} = 0; M_{36} = 0; \\
 M_{37} &= 0; M_{41} = [-I_6\beta]; M_{42} = 0; M_{43} = 0; \\
 M_{44} &= [I_9]; M_{45} = [I_{10}]; M_{46} = 0; M_{47} = 0; \\
 M_{51} &= [-I_7\beta]; M_{52} = 0; M_{53} = 0; \\
 M_{54} &= [I_{10}]; M_{55} = [I_{15}]; M_{56} = 0; M_{57} = 0; \\
 M_{61} &= [I_{16}]; M_{62} = 0; M_{63} = 0; M_{64} = 0; \\
 M_{65} &= 0; M_{66} = [I_{18}]; M_{67} = [I_{19}]; \\
 M_{71} &= [I_{17}]; M_{72} = 0; M_{73} = 0; M_{74} = 0; \\
 M_{75} &= 0; M_{76} = [I_{19}]; M_{77} = [I_{20}];
 \end{aligned} \tag{A-1}$$

The elements of stiffness matrix [K] are as follows:

$$\begin{aligned}
 K_{11} &= [D_1 \alpha^4 + (2D_2 + 4D_3) \alpha^2 \beta^2 + D_1 \beta^2]; \\
 K_{12} &= [-L_1 \alpha^3 + (L_2 + 2L_3) \alpha \beta^2]; \\
 K_{13} &= [-M_1 \alpha^3 + (M_2 + 2M_3) \alpha \beta^2]; \\
 K_{14} &= [-L_1 \beta^3 + (L_2 + 2L_3) \alpha^2 \beta]; \\
 K_{15} &= [-M_1 \beta^3 + (M_2 + 2M_3) \alpha^2 \beta]; \\
 K_{16} &= -\frac{8}{h^2} [D_2 \alpha^2 + D_2 \beta^2]; \\
 K_{17} &= -\frac{64}{h^4} [F_2 \alpha^2 + F_2 \beta^2]; \\
 K_{21} &= [-L_1 \alpha^3 + (L_2 + 2L_3) \alpha \beta^2]; \\
 K_{22} &= [(N_1 \alpha^2 + N_4 \beta^2) + C_3]; \\
 K_{23} &= [(P_1 \alpha^2 + P_3 \beta^2) + H_3]; \\
 K_{24} &= [(N_2 + N_3) \alpha \beta]; \\
 K_{25} &= [(P_2 + P_3) \alpha \beta]; \\
 K_{26} &= \left[\left(\frac{8}{h^2} L_2 + C_3 \right) \alpha \right]; \\
 K_{27} &= \left[\left(\frac{64}{h^4} S_2 + H_3 \right) \alpha \right]; \\
 K_{31} &= [-MA_1 \alpha^3 + (MA_2 + 2MA_3) \alpha \beta^2]; \\
 K_{32} &= [(P_1 \alpha^2 + P_3 \beta^2) + H_3]; \\
 K_{33} &= [(R_1 \alpha^2 + R_3 \beta^2) + G_3]; \\
 K_{34} &= [(P_2 + P_3) \alpha \beta]; \\
 K_{35} &= [(R_2 + R_3) \alpha \beta]; \\
 K_{36} &= \left[\left(\frac{8}{h^2} MA_2 + H_3 \right) \alpha \right]; \\
 K_{37} &= \left[\left(\frac{64}{h^4} T_2 + G_3 \right) \alpha \right]; \\
 K_{41} &= [-L_1 \beta^3 + (L_2 + 2L_3) \alpha^2 \beta]; \\
 K_{42} &= [(N_2 + N_3) \alpha \beta]; \\
 K_{43} &= [(P_2 + P_3) \alpha \beta]; \\
 K_{44} &= [(N_1 \beta^2 + N_3 \alpha^2) + C_3]; \\
 K_{45} &= [(P_1 \beta^2 + P_3 \alpha^2) + H_3]; \\
 K_{46} &= \left[\left(\frac{8}{h^2} L_2 + C_3 \right) \beta \right]; \\
 K_{47} &= \left[\left(\frac{64}{h^4} S_2 + H_3 \right) \beta \right]; \\
 K_{51} &= [-MA_1 \beta^3 + (MA_2 + 2MA_3) \alpha^2 \beta]; \\
 K_{52} &= [(P_2 + P_3) \alpha \beta]; \\
 K_{53} &= [(R_2 + R_3) \alpha \beta]; \\
 K_{54} &= [(P_1 \beta^2 + P_3 \alpha^2) + H_3]; \\
 K_{55} &= [(R_1 \beta^2 + R_3 \alpha^2) + G_3]; \\
 K_{56} &= \left[\left(\frac{8}{h^2} MA_2 + H_3 \right) \beta \right]; \\
 K_{57} &= \left[\left(\frac{64}{h^4} T_2 + G_3 \right) \beta \right]; \\
 K_{61} &= -\left[\frac{8}{h^2} (D_2 \alpha^2 + D_2 \beta^2) \right]; \\
 K_{62} &= \left[\left(\frac{8}{h^2} L_2 + C_3 \right) \alpha \right]; \\
 K_{63} &= \left[\left(\frac{8}{h^2} MA_2 + H_3 \right) \alpha \right]; \\
 K_{64} &= \left[\left(\frac{8}{h^2} L_2 + C_3 \right) \beta \right]; K_{65} = \left[\left(\frac{8}{h^2} M_2 + H_3 \right) \beta \right]; \\
 K_{66} &= [C_3 \alpha^2 + C_3 \beta^2 + D_1 \frac{64}{h^4}]; K_{67} = [H_3 \alpha^2 + H_3 \beta^2 + F_1 \frac{512}{h^6}]; \\
 K_{71} &= -\left[\frac{64}{h^4} (F_2 \alpha^2 + F_2 \beta^2) \right];
 \end{aligned}$$

$$\begin{aligned}
K_{72} &= \left[\left(\frac{64}{h^4} S_2 + H_3 \right) \alpha \right]; K_{73} = \left[\left(\frac{64}{h^4} T_2 + G_3 \right) \alpha \right]; \\
K_{74} &= \left[\left(\frac{64}{h^4} S_2 + H_3 \right) \beta \right]; K_{75} = \left[\left(\frac{64}{h^4} T_2 + G_3 \right) \beta \right]; \\
K_{76} &= \left[H_3 \alpha^2 + H_3 \beta^2 + F_1 \frac{512}{h^6} \right]; \\
K_{77} &= \left[G_3 \alpha^2 + G_3 \beta^2 + O_1 \frac{4096}{h^8} \right];
\end{aligned}$$

The functional operators L_{11} etc are as follows:

$$\begin{aligned}
L_{11} &= D_1(\cdot)_{,xxxx} + (2D_2 + 4D_3)(\cdot)_{,xxyy} \\
&\quad + D_1(\cdot)_{,yyyy} - I_5(\cdot)_{,xxtt} - I_5(\cdot)_{,yytt} - I_1(\cdot)_{,tt} \\
L_{12} &= -L_1(\cdot)_{,xxx} - (L_2 + 2L_3)(\cdot)_{,xyy} + I_6(\cdot)_{,xtt} \\
L_{13} &= -MA_1(\cdot)_{,xxx} - (MA_2 + 2MA_3)(\cdot)_{,xyy} + I_7(\cdot)_{,xtt} \\
L_{14} &= -L_1(\cdot)_{,yyy} - (L_2 + 2L_3)(\cdot)_{,xxy} + I_6(\cdot)_{,ytt} \\
L_{15} &= -MA_1(\cdot)_{,yyy} - (MA_2 + 2MA_3)(\cdot)_{,xxy} + I_7(\cdot)_{,ytt} \\
L_{16} &= -D_2 \frac{8}{h^2} \left((\cdot)_{,xx} + (\cdot)_{,yy} \right) + I_{16}(\cdot)_{,tt} \\
L_{17} &= -F_2 \frac{64}{h^4} \left((\cdot)_{,xx} + (\cdot)_{,yy} \right) + I_{17}(\cdot)_{,tt} \\
L_{21} &= L_1(\cdot)_{,xxx} + (L_2 + 2L_3)(\cdot)_{,xyy} - I_6(\cdot)_{,xtt} \\
L_{22} &= -N_1(\cdot)_{,xx} - N_3(\cdot)_{,yy} + C_3(\cdot)_{,x} + I_9(\cdot)_{,tt} \\
L_{23} &= -P_1(\cdot)_{,xx} - P_3(\cdot)_{,yy} + H_3(\cdot)_{,x} + I_{10}(\cdot)_{,tt} \\
L_{24} &= -(N_2 + N_3)(\cdot)_{,xy} \\
L_{25} &= -(P_2 + P_3)(\cdot)_{,xy} \\
L_{26} &= \left(\frac{8}{h^2} L_2 + C_3 \right) (\cdot)_{,x} \\
L_{27} &= \left(\frac{64}{h^4} S_2 + H_3 \right) (\cdot)_{,x} \\
L_{31} &= MA_1(\cdot)_{,xxx} + (MA_2 + 2MA_3)(\cdot)_{,xyy} - I_7(\cdot)_{,xtt} \\
L_{32} &= -P_1(\cdot)_{,xx} - P_3(\cdot)_{,yy} + H_3(\cdot)_{,x} + I_{10}(\cdot)_{,tt} \\
L_{33} &= -R_1(\cdot)_{,xx} - R_3(\cdot)_{,yy} + GA_3(\cdot)_{,x} + I_{15}(\cdot)_{,tt} \\
L_{34} &= -(P_2 + P_3)(\cdot)_{,xy} \\
L_{35} &= -(R_2 + R_3)(\cdot)_{,xy} \\
L_{36} &= \left(\frac{8}{h^2} MA_2 + H_3 \right) (\cdot)_{,x} \\
L_{37} &= \left(\frac{64}{h^4} T_2 + GA_3 \right) (\cdot)_{,x} \\
L_{41} &= L_1(\cdot)_{,yyy} + (L_2 + 2L_3)(\cdot)_{,xxy} - I_6(\cdot)_{,ytt}
\end{aligned}$$

$$\begin{aligned}
L_{42} &= -(N_2 + N_3)(\cdot)_{,xy} \\
L_{43} &= -(P_2 + P_3)(\cdot)_{,xy} \\
L_{44} &= -N_1(\cdot)_{,yy} - N_3(\cdot)_{,xx} + C_3(\cdot)_{,y} + I_9(\cdot)_{,tt} \\
L_{45} &= -P_1(\cdot)_{,yy} - P_3(\cdot)_{,xx} + H_3(\cdot)_{,y} + I_{10}(\cdot)_{,tt} \\
L_{46} &= \left(\frac{8}{h^2} L_2 + C_3 \right) (\cdot)_{,y} \\
L_{47} &= \left(\frac{64}{h^4} S_2 + H_3 \right) (\cdot)_{,y} \\
L_{51} &= MA_1(\cdot)_{,yyy} + (MA_2 + 2MA_3)(\cdot)_{,xxy} - I_7(\cdot)_{,ytt} \\
L_{52} &= -(P_2 + P_3)(\cdot)_{,xy} \\
L_{53} &= -(R_2 + R_3)(\cdot)_{,xy} \\
L_{54} &= -P_1(\cdot)_{,yy} - P_3(\cdot)_{,xx} + H_3 + I_{10}(\cdot)_{,tt} \\
L_{55} &= -R_1(\cdot)_{,yy} - R_3(\cdot)_{,xx} + GA_3 + I_{15}(\cdot)_{,tt} \\
L_{56} &= \left(\frac{8}{h^2} MA_2 + H_3 \right) (\cdot)_{,y} \\
L_{57} &= \left(\frac{64}{h^4} T_2 + GA_3 \right) (\cdot)_{,y} \\
L_{61} &= \frac{8}{h^2} D_2 \left((\cdot)_{,xx} + (\cdot)_{,yy} \right) + I_{16}(\cdot)_{,tt} \\
L_{62} &= -\left(\frac{8}{h^2} L_2 + C_3 \right) (\cdot)_{,x} \\
L_{63} &= -\left(\frac{8}{h^2} MA_2 + H_3 \right) (\cdot)_{,x} \\
L_{64} &= -\left(\frac{8}{h^2} L_2 + C_3 \right) (\cdot)_{,y} \\
L_{65} &= -\left(\frac{8}{h^2} MA_2 + H_3 \right) (\cdot)_{,y} \\
L_{66} &= -C_3 \left((\cdot)_{,yy} + (\cdot)_{,xx} \right) + D_1 \frac{64}{h^4} + I_{18}(\cdot)_{,tt} \\
L_{67} &= -H_3 \left((\cdot)_{,yy} + (\cdot)_{,xx} \right) + F_1 \frac{512}{h^6} + I_{19}(\cdot)_{,tt} \\
L_{71} &= \frac{64}{h^4} F_2 \left((\cdot)_{,xx} + (\cdot)_{,yy} \right) + I_{17}(\cdot)_{,tt} \\
L_{72} &= -\left(\frac{64}{h^4} S_2 + H_3 \right) (\cdot)_{,x} \\
L_{73} &= -\left(\frac{64}{h^4} T_2 + GA_3 \right) (\cdot)_{,x} \\
L_{74} &= -\left(\frac{64}{h^4} S_2 + H_3 \right) (\cdot)_{,y} \\
L_{75} &= -\left(\frac{64}{h^4} T_2 + GA_3 \right) (\cdot)_{,y} \\
L_{76} &= -H_3 \left((\cdot)_{,yy} + (\cdot)_{,xx} \right) + F_1 \frac{512}{h^6} + I_{19}(\cdot)_{,tt} \\
L_{77} &= -GA_3 \left((\cdot)_{,yy} + (\cdot)_{,xx} \right) + O_1 \frac{4096}{h^8} + I_{20}(\cdot)_{,tt}
\end{aligned}$$

The functional operators L_{11}^0 etc are as follows:

$$L_{11}^0 = -D_1(\cdot)_{,xxx} - (D_2 + 4D_3)(\cdot)_{,xyy} + I_5(\cdot)_{,xtt}$$

$$L_{12}^0 = L_1(\cdot)_{,xx} + (2L_3)(\cdot)_{,yy} - I_6(\cdot)_{,tt}$$

$$L_{13}^0 = MA_1(\cdot)_{,xx} + (2MA_3)(\cdot)_{,yy} + I_7(\cdot)_{,tt}$$

$$L_{14}^0 = (L_2 + 2L_3)(\cdot)_{,xy}$$

$$L_{15}^0 = (MA_2 + 2MA_3)(\cdot)_{,xy}$$

$$L_{16}^0 = -D_2 \frac{8}{h^2}(\cdot)_{,x}$$

$$L_{17}^0 = -F_2 \frac{64}{h^4}(\cdot)_{,x}$$

$$L_{21}^0 = D_1(\cdot)_{,xx} + D_2(\cdot)_{,yy}$$

$$L_{22}^0 = -L_1(\cdot)_{,x}$$

$$L_{23}^0 = -MA_1(\cdot)_{,x}$$

$$L_{24}^0 = -L_2(\cdot)_{,y}$$

$$L_{25}^0 = -MA_2(\cdot)_{,y}$$

$$L_{26}^0 = D_2 \frac{8}{h^2}$$

$$L_{27}^0 = F_2 \frac{64}{h^4}$$

$$L_{31}^0 = -L_1(\cdot)_{,xx} - L_2(\cdot)_{,yy}$$

$$L_{32}^0 = N_1(\cdot)_{,x}$$

$$L_{33}^0 = P_1(\cdot)_{,x}$$

$$L_{34}^0 = N_2(\cdot)_{,y}$$

$$L_{35}^0 = P_2(\cdot)_{,y}$$

$$L_{36}^0 = -L_2 \frac{8}{h^2}$$

$$L_{37}^0 = -S_2 \frac{64}{h^4}$$

$$L_{41}^0 = -MA_1(\cdot)_{,xx} - MA_2(\cdot)_{,yy}$$

$$L_{42}^0 = P_1(\cdot)_{,x}$$

$$L_{43}^0 = R_1(\cdot)_{,x}$$

$$L_{44}^0 = P_2(\cdot)_{,y}$$

$$L_{45}^0 = R_2(\cdot)_{,y}$$

$$L_{46}^0 = -MA_2 \frac{8}{h^2}$$

$$L_{47}^0 = -T_2 \frac{64}{h^4}$$

$$L_{51}^0 = -2L_3(\cdot)_{,xy}$$

$$L_{52}^0 = N_3(\cdot)_{,y}$$

$$L_{53}^0 = P_3(\cdot)_{,y}$$

$$L_{54}^0 = N_3(\cdot)_{,x}$$

$$L_{55}^0 = P_3(\cdot)_{,x}$$

$$L_{56}^0 = L_{57}^0 = 0$$

$$L_{61}^0 = -2MA_3(\cdot)_{,xy}$$

$$L_{62}^0 = P_3(\cdot)_{,y}$$

$$L_{63}^0 = R_3(\cdot)_{,y}$$

$$L_{64}^0 = P_3(\cdot)_{,x}$$

$$L_{65}^0 = R_3(\cdot)_{,x}$$

$$L_{66}^0 = L_{67}^0 = 0$$

$$L_{71}^0 = L_{72}^0 = L_{73}^0 = 0$$

$$L_{74}^0 = C_3$$

$$L_{75}^0 = H_3$$

$$L_{76}^0 = C_3(\cdot)_{,y}$$

$$L_{77}^0 = H_3(\cdot)_{,y}$$

$$L_{81}^0 = L_{82}^0 = L_{83}^0 = 0$$

$$L_{84}^0 = H_3$$

$$L_{85}^0 = G_3$$

$$L_{86}^0 = H_3(\cdot)_{,y}$$

$$L_{87}^0 = G_3(\cdot)_{,y}$$

The functional operators L_{11}^0 etc are as follows:

$$L_{S11} = -D_1(\cdot)_{,yyy} - (D_2 + 4D_3)(\cdot)_{,xxy} + I_5(\cdot)_{,ytt}$$

$$L_{S12} = (L_2 + 2L_3)(\cdot)_{,xy}$$

$$L_{S13} = (MA_2 + 2MA_3)(\cdot)_{,xy}$$

$$L_{S14} = L_1(\cdot)_{,yy} + (2L_3)(\cdot)_{,xx} - I_6(\cdot)_{,tt}$$

$$L_{S15} = MA_1(\cdot)_{,yy} + (2MA_3)(\cdot)_{,xx} + I_7(\cdot)_{,tt}$$

$$L_{S16} = -D_2 \frac{8}{h^2}(\cdot)_{,y}$$

$$L_{S17} = -F_2 \frac{64}{h^4}(\cdot)_{,y}$$

$$L_{S21} = D_2(\cdot)_{,xx} + D_1(\cdot)_{,yy}$$

$$L_{S22} = -L_2(\cdot)_{,x}$$

$$L_{S23} = -MA_2(\cdot)_{,x}$$

$$\begin{aligned}
L_{S24} &= -L_1(\cdot)_{,y} \\
L_{S25} &= -MA_1(\cdot)_{,y} \\
L_{S26} &= D_2 \frac{8}{h^2} \\
L_{S27} &= F_2 \frac{64}{h^4} \\
L_{S31} &= -2L_3(\cdot)_{,xy} \\
L_{S32} &= N_3(\cdot)_{,y} \\
L_{S33} &= P_3(\cdot)_{,y} \\
L_{S34} &= N_3(\cdot)_{,x} \\
L_{S35} &= P_3(\cdot)_{,x} \\
L_{S36} &= L_{S37} = 0 \\
L_{S41} &= -2MA_3(\cdot)_{,xy} \\
L_{S42} &= P_3(\cdot)_{,y} \\
L_{S43} &= R_3(\cdot)_{,y} \\
L_{S44} &= P_3(\cdot)_{,x} \\
L_{S45} &= R_3(\cdot)_{,x} \\
L_{S46} &= L_{S47} = 0 \\
L_{S51} &= -L_2(\cdot)_{,xx} - L_1(\cdot)_{,yy} \\
L_{S52} &= N_2(\cdot)_{,x} \\
L_{S53} &= P_2(\cdot)_{,x} \\
L_{S54} &= N_1(\cdot)_{,y} \\
L_{S55} &= P_1(\cdot)_{,y} \\
L_{S56} &= -L_2 \frac{8}{h^2} \\
L_{S57} &= -S_2 \frac{64}{h^4} \\
L_{S61} &= -MA_1(\cdot)_{,xx} - MA_2(\cdot)_{,yy} \\
L_{S62} &= P_2(\cdot)_{,x} \\
L_{S63} &= R_2(\cdot)_{,x} \\
L_{S64} &= P_1(\cdot)_{,y} \\
L_{S65} &= R_1(\cdot)_{,y} \\
L_{S66} &= -MA_2 \frac{8}{h^2} \\
L_{S67} &= -T_2 \frac{64}{h^4} \\
L_{S71} &= L_{S72} = L_{S73} = 0 \\
L_{S74} &= C_3 \\
L_{S75} &= H_3 \\
L_{S76} &= C_3(\cdot)_{,y} \\
L_{S77} &= H_3(\cdot)_{,y} \\
L_{S81} &= L_{S82} = L_{S83} = 0 \\
L_{S84} &= H_3 \\
L_{S85} &= G_3 \\
L_{S86} &= H_3(\cdot)_{,y} \\
L_{S87} &= G_3(\cdot)_{,y} \\
L_{P11} &= 0 \\
L_{P12} &= C_3(\cdot)_{,y} \\
L_{P13} &= D_3(\cdot)_{,y} \\
L_{P14} &= C_3(\cdot)_{,x} \\
L_{P15} &= D_3(\cdot)_{,x} \\
L_{P16} &= L_{P17} = 0
\end{aligned}$$

In these equation, $(\cdot)_{,x}$ and $(\cdot)_{,y}$, respectively, denote derivation with respect to x and y , respectively, while $(\cdot)$ indicates a derivation with respect to time t .

The constant appeared in the governing equations and boundary conditions are as follows:

$$\begin{aligned}
(I_1, I_{16}, I_{17}, I_{18}, I_{19}, I_{20}) &= \rho h \left(1, \frac{2}{3}, \frac{4}{5}, \frac{8}{15}, \frac{64}{105}, \frac{32}{45} \right) \\
(I_5, I_6, I_7, I_9, I_{10}, I_{15}) &= \rho h^3 \left(\frac{1}{12}, \frac{1}{15}, \frac{8}{105}, \frac{17}{315}, \frac{58}{945}, \frac{404}{5775} \right) \\
\left\{ \begin{matrix} D_1, F_1, L_1, MA_1, \\ NA_1, S_1, P_1, R_1, T_1, O_1 \end{matrix} \right\} &= (\lambda + 2G) \left\{ \frac{h^3}{12}, \frac{h^5}{80}, \frac{h^3}{15}, \frac{8h^3}{105}, \frac{17h^3}{315}, \frac{h^5}{105}, \frac{58h^3}{945}, \frac{404h^3}{5775}, \frac{h^5}{90}, \frac{h^7}{448} \right\} \\
\left\{ \begin{matrix} D_2, F_2, L_2, MA_2, \\ NA_2, S_2, P_2, R_2, T_2 \end{matrix} \right\} &= (\lambda) \left\{ \frac{h^3}{12}, \frac{h^5}{80}, \frac{h^3}{15}, \frac{8h^3}{105}, \frac{17h^3}{315}, \frac{h^5}{105}, \frac{58h^3}{945}, \frac{404h^3}{5775}, \frac{h^5}{90} \right\} \\
\left\{ \begin{matrix} C_3, D_3, F_3, GA_3, H_3, L_3, \\ MA_3, NA_3, S_3, P_3, R_3, T_3, O_3 \end{matrix} \right\} &= (G) \left\{ \frac{8h}{15}, \frac{h^3}{12}, \frac{h^5}{80}, \frac{64h}{105}, \frac{h^3}{15}, \frac{8h^3}{105}, \frac{17h^3}{315}, \frac{h^5}{105}, \frac{58h^3}{945}, \frac{404h^3}{5775}, \frac{h^5}{90}, \frac{h^7}{448} \right\}
\end{aligned} \tag{A-6}$$

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