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Free vibration analysis of functionally graded sandwich plates using a new discrete kirchhoff four node quadrilateral element

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Abstract

In the present work, a four node quadrilateral element is developed for the free vibration analysis of functionally graded moderately thick sandwich plates. The element has seven degrees of freedom per node namely, two in-plane displacements, the transverse displacement, two rotations of mid plane and two transverse shear strain components at the mid plane. In the present study, two models of the material property variation through the thickness, namely; power law model and Mori-Tanaka model are considered. The results for sandwich plates with different thickness configurations and with different boundary conditions are compared with the available analytical and finite element results. For some cases, the present results of the non-dimensionalised frequencies are compared with those obtained using commercial software ANSYS. It is observed that the performance of the present element is quite satisfactory for the sandwich plates considered in this study.

Keywords: functionally graded material, finite element, quadrilateral element, third order theory

1. Introduction

Functionally graded materials (FGM) are being increasingly used in aeronautical and aerospace industry as well as in other fields of modern technology, especially where structures are subjected to high temperatures. Functionally graded materials normally consist of two phases, namely; the metal phase and the ceramic phase. Ceramic acts as thermal barrier and the metal give required ductility. The FGM have advantage over the laminated composites. The delamination problem that is normally observed in laminated composites is totally absent in FGM because of the smooth variation of material properties through the thickness. The variation of the material property through the thickness can be predicted either by the widely used simple rule of mixtures (ROM) model or by Mori-Tanaka (MT) model. The Mori-Tanaka model presented by Mori and Tanaka [1] accounts for the interaction of the elastic fields of neighboring inclusions, and yields more accurate prediction of the material properties through the thickness than the ROM model. But as the rule of mixture model is simple to use, many researchers are still using it and hence this model is also considered in the present study.
The FGM structures are normally modeled as plates and shells and determination of their natural frequencies is essential in many applications. Many researchers have presented 3 D exact as well as 3 D approximate solutions for the free vibration response of functionally graded (FG) plates. 3 D exact solution is presented for free and forced vibration of simply supported functionally graded rectangular plates by Vel and Batra [2]. Results are also computed for a functionally graded plate that has a varying microstructure in the thickness direction using a combination of the Mori-Tanaka and the self-consistent methods. 3 D elasticity solution using the Ritz method was developed by Li et al. [3] for the free vibration analysis of symmetric and unsymmetric FGM sandwich plates with simply-supported and clamped boundary conditions. The study shows that the convergence is independent of material gradient parameter. Many researchers have also developed 2 D analytical solutions based on 2 D theories. Kalhori et al. [4] have analyzed FG rectangular plates based on Mori-Tanaka homogenization scheme for vibration response using first order shear deformation theory (FSDT). A trigonometric shear deformation plate theory which considers the effect of rotary inertia is used by Zenkour [5] for investigating natural frequencies of simply supported FG thick plates. Zenkour [6] has also studied buckling and free vibration analysis of functionally graded sandwich plates using sinusoidal shear deformation plate theory. Exact free vibration analysis of Levy FG plates using higher-order shear and normal deformation theories is given by Dozio [7].
2 D analytical solutions also are not available for complex geometries and for complicated boundary conditions, therefore many researchers have employed 2 D finite element technique. Natarajan and Manickam [8] used an eight-node finite element with 13 DOF per node based on their accurate theory for bending and vibration analysis of FG sandwich plates. A continuous Lagrangian quadrilateral

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nine-noded element with eight degrees of freedom per node based on a new hyperbolic higher-order shear and normal deformation theory (HHSNDT) is employed by Gupta and Talha [9] for the large amplitude vibration analysis of FG plates. An eight-noded C^0 isoparametric element with 13 degrees of freedom per node based on Layerwise theory is used by Pandey and Pradyumna [10] to develop the finite element model to determine thermally induced vibrations of FG sandwich plates and shell panels.
Reddy's third order shear deformation theory (TOT) [11] is one of the best available 2 D theories for the analysis of moderately thick FG plates but the finite element formulation based on this theory poses a problem of C^1 continuity. Thus it is difficult to develop a four node quadrilateral element based on Reddy's theory. The present second author along with his co-author [12, 13] earlier developed a four node discrete Kirchhoff quadrilateral element for the analysis of laminated composite and sandwich plates wherein the problem of C^1 continuity was successfully circumvented. The second author and his co-workers [14] extended the concept of [12, 13] and developed a four node discrete Kirchhoff quadrilateral element based on Reddy's third order theory for static and free vibration analysis of functionally graded single layer skew plates. The present authors by modifying the four node discrete Kirchhoff quadrilateral element of [14] have analysed a single layer FG plate for free vibration response under thermal environment [15]. In the present work the four node discrete Kirchhoff quadrilateral element of [14] is modified for the analysis of FG sandwich plates.

2. Formulation

2.1 Displacement Field Approximation for Reddy's Third Order Theory

Consider a sandwich plate (Fig. 1) of total thickness 'H' with the top and the bottom layers made of functionally graded material. The middle layer is of ceramic. Mid surface of the plate is considered as reference plane where 'z = 0' and the top and the bottom surfaces are at 'z= H/2' and 'z= -H/2', respectively.
The functionally graded sandwich plates considered in this study are:
Type 1: Plates where variation of material properties is as per power law (ROM model):

$$P(z) = P_b + (P_t - P_b)\left[\frac{2z+h}{2h}\right]^n \quad (1)$$

where 'n' is the volume fraction index, 'P_b' and 'P_t' are the material properties at the bottom and at the top of the layer respectively and 'h' is the thickness of the layer. Poisson's ratio is assumed to be same for both the materials**.**
Type 2: Plates where variation of material properties is as per Mori- Tanaka model of [1]

$$\frac{K-K_b}{K_t-K_b} = \frac{V_c}{1+(1-V_c)\dfrac{K_t-K_b}{K_b+4/3G_b}}; \quad and$$
$$\frac{G-G_b}{G_t-G_b} = \frac{V_c}{1+(1-V_c)\dfrac{G_t-G_b}{G_b+f_1}} \quad (2)$$

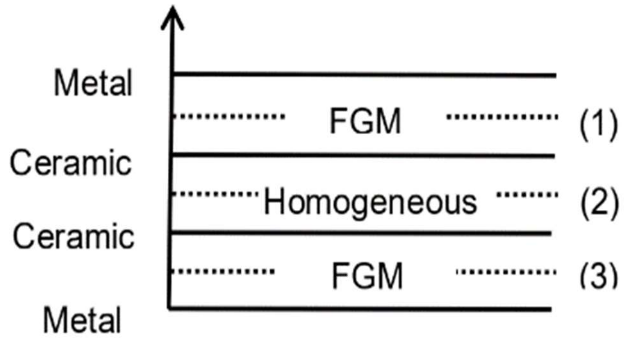

Fig. 1. FGM Sandwich Plate

$$f_1 = \frac{G_b(9K_b+8G_b)}{6(K_b+2G_b)} \quad (3)$$
$$V_c = (0.5+\frac{z}{h})^n \quad (4)$$
$$E = \frac{9KG}{3K+G}; \ \upsilon = \frac{3K-2G}{2(3K+G)} \quad (5)$$

where,
E = The effective value of Young's modulus at any surface at z in the layer; υ = The effective value of Poisson's ratio at any surface at z in the layer; K_b= bulk modulus of the material at the bottom of the layer; K_t = bulk modulus of the material at the top of the layer; G_b = shear modulus of the material at the bottom of the layer and G_t = shear modulus of the material at the top of the layer.
The effective mass density ρ at any surface at z in the layer is obtained using power law as per equation (1).
In the third order theory of Reddy [11], the deflection 'w' is approximated to be independent of 'z' and the in plane displacements u_x , u_y are approximated as a cubic variation in 'z' across the thickness:

$$w(x,y,z,t) = w_0(x,y,t)$$
$$u(x,y,z,t) = u_0(x,y,t) - zw_{0d} + R(z)\psi_0(x,y,t) \quad (6)$$

Where $R(z) = z - \frac{4z^3}{3h^2}$
The constitutive relations σ can be expressed as

$$\sigma = \bar{Q}_{ij}[\varepsilon] \quad (7)$$

Stiffness coefficients $\bar{Q}_{ij}$ are defined as:

$$\bar{Q}_{11} = \bar{Q}_{22} = \frac{E(z)}{1-\upsilon^2}; \ \bar{Q}_{12} = \bar{Q}_{21} = \frac{\upsilon E(z)}{1-\upsilon^2};$$
$$\bar{Q}_{16} = \bar{Q}_{26} = 0; \ \bar{Q}_{44} = \bar{Q}_{55} = \bar{Q}_{66} = \frac{E(z)}{2(1+\upsilon)} \quad (8)$$

2.2 Finite Element Formulation

Principle of virtual work is used for obtaining stiffness matrix and mass matrix. Details regarding the interpolation functions are given in [14].

3. Numerical Results and Discussions

The performance of the four node quadrilateral element developed in earlier section is tested for its efficiency for free vibration response by analysing the two types of sandwich plates the material properties of which are given in Table 1.

Table 1.Metals and Ceramics with their properties.

Material	E(Gpa)	μ	P(kg/m^3)
Aluminum (metal) (*Al*)	70	0.3	2707
Zircon (Ceramic) (ZrO_2)	200	0.3	5700
Alumina (Ceramic)(Al_2O_3)	380	0.3	3800

The results of present element for non- dimensionalised frequencies obtained by discretizing the full plate with following three mesh sizes $N\ X\ N = 12\ X\ 12; 16\ X\ 16$, and $20\ X\ 20$ are compared with the available analytical results and with the available finite element results based on various theories. In some cases present element results have also been compared with that obtained by commercial software ANSYS [16]. In ANSYS the facility for providing material properties of FG plate is not available hence FG plate is considered as a layered plate comprising of suitable number of layers. The material properties are obtained at the centre of each layer and considered to be constant over that layer. If a specific formula is not given for the non-dimensionalised frequency then, it is obtained as follows,

$$\bar{\omega} = \frac{\omega a^2}{H}\sqrt{\frac{\rho_0}{E_0}}$$

Where,

a = shorter dimension of the plate, *H* = total thickness of the plate, $E_0 = 1GPa$, $\rho_0 = 1kg/m^3$.
Materials properties are given in Table 1.

3.1 Convergence Study for Type 1 plate

The results of all round simply supported square FG sandwich plates with *a/h* = 10 for thickness configurations, 1-2-1 and 2-1-1 are presented in tables 2 and 3 respectively for various volume fractions. The present results are compared with the analytical results based on TOT as given by Zenkour [6]. The present results with mesh size 20 X 20 are closely matching with the analytical results given in Zenkour [6]. It is also observed that as the mesh size increases the results converge to the analytical results of TOT for all configurations, establishing the correctness of the present formulation.
The first six mode shapes for an all round simply supported square sandwich plate with thickness configuration 1-2-1 and with *a=h* = 10 for volume fraction index *n* = 2 obtained with a mesh size 20 X 20 are shown in (Fig. 2). Non-dimensionalised frequencies for all the six modes are matching with that of [6]. It is also observed that 2 D analytical solution of [6] cannot capture the shear modes, which the present formulation can capture.

3.2. Type 2 Plate

Since the results for Type 2 sandwich plate of the best of knowledge of the authors, are not available for comparison, a single layer FG plate is analysed with present formulation and the results are compared with the analytical results based on the higher order shear and normal deformation theory (HOSNT) of Dozio [7] to assess the accuracy of the present formulation.

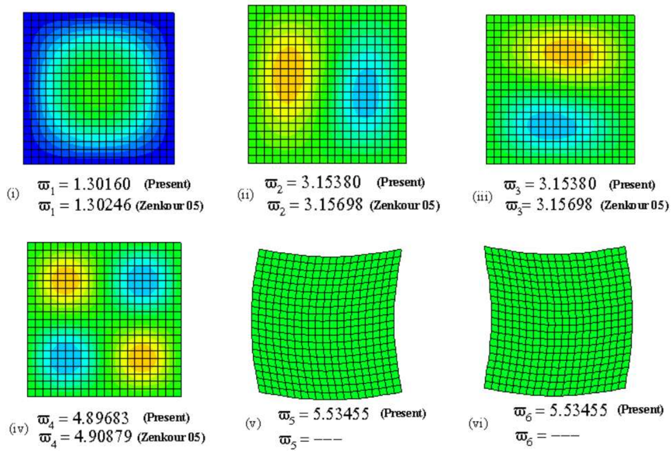

Fig. 2. First six mode shapes for all round simply supported square Type 1 sandwich plate with thickness configuration 1-2-1 and with *a=h* = 10 for volume fraction index *n* = 2 (*Al/Al2O3*)

Table 2. Non-dimensionalised fundamental frequency $\bar{\omega}$ for square plate A with thickness configuration 1-2-1

Volume fraction	(Zenkour,2005)	Present FE results Mesh Size		
n	ϖ	12 × 12	16 × 16	20 × 20
0	1.82445	1.82115	1.82258	1.82325
0.5	1.57451	1.57182	1.57305	1.57363
1	1.43934	1.43673	1.43786	1.43839
5	1.17397	1.17185	1.17277	1.17320
10	1.12314	1.12111	1.12199	1.12240

Table 3. Non-dimensionalised fundamental frequency $\bar{\omega}$ for square plate A with thickness configuration 2-1-1

Volume fraction	(Zenkour,2005)	Present FE results Mesh Size		
n	ϖ	12 × 12	16 × 16	20 × 20
0	1.82445	1.82115	1.82258	1.82325
0.5	1.51253	1.50395	1.50513	1.50568
1	1.34888	1.33104	1.33206	1.33254
5	1.07432	1.02887	1.02962	1.02996
10	1.03862	0.99047	0.99118	0.99151

The material property variation through the thickness is as per MT model. Once it is assessed, the results for Type 2 sandwich plate are also given for various thickness configurations.

3.2.1. Results for single layer FG plate

In Table 4 results of the non-dimensionalised natural frequencies for first five modes for an all-round simply supported square single layer plate with *a/h* = 5 and 10 for *n* = 1 are compared with the analytical results based on HOSNT presented by Dozio [7]. It is observed that the present results are quite close to the analytical results of [7] even for higher modes. In Table 5 the non-dimensionalised fundamental frequency for square plate, with *a/h* = 10 and with different boundary conditions are compared with the 2D analytical results of [7]. For this case also the present results are quite close to that of [7]. The results based on ROM model are also given for comparison in Tables 5 and it is observed from the Table 5 that the results of the non-dimensionalised frequencies obtained by MT model are lower than that obtained by ROM model.

Table 4. First six non dimensionalised frequencies for all round simply supported single layer plate ($Al=ZrO2$)

a/h	Results	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
5	HOSNT-9	0.2213	0.4116	0.4116	0.4906	0.4906
	HOSNT-12	0.2193	0.4116	0.4116	0.4824	0.4824
	HOSNT-15	0.2191	0.4116	0.4116	0.4820	0.4820
	Present (12X12)	0.2180	0.4127	0.4127	0.4781	0.4781
	Present (16X16)	0.2182	0.4122	0.4122	0.4785	0.4785
	Present (20X20)	0.2184	0.4122	0.4122	0.4700	0.4700
10	HOSNT-9	0.0597	0.1435	0.1435	0.2059	0.2059
	HOSNT-12	0.0596	0.1425	0.1425	0.2059	0.2059
	HOSNT-15	0.0596	0.1425	0.1425	0.2059	0.2059
	Present (12X12)	0.0594	0.1418	0.1418	0.2064	0.2064
	Present (16X16)	0.0595	0.1420	0.1420	0.2061	0.2061
	Present (20X20)	0.0595	0.1420	0.1420	0.2061	0.2061

Table 5.Non-dimensionalised fundamental frequency for single layer plate with different boundary conditions ($Al=ZrO2$)

n	Results	SSSC	SCSC
1	HOSNT-9	0.0705	0.0845
	HOSNT-12	0.0703	0.0839
	HOSNT-15	0.0701	0.0839
	Present (12X12)	0.0700	0.0838
		(0.0729)	(0.0873)
	Present (16X16)	0.0700	0.0837
		(0.0729)	(0.0872)
	Present (20X20)	0.0700	0.0836
		(0.0729)	(0.0872)
2	HOSNT-9	0.0711	0.0851
	HOSNT-12	0.0708	0.0843
	HOSNT-15	0.0706	0.0843
	Present (12X12)	0.0705	0.0842
		(0.0725)	(0.0868)
	Present (16X16)	0.0705	0.0841
		(0.0726)	(0.0867)
	Present (20X20)	0.0705	0.0841
		(0.0726)	(0.0866)

Values of the non-dimensionalised frequencies in the bracket are for the same plate with ROM model

Table 6: First six non-dimensionalised frequencies for all round simply supported square Type 2 sandwich plate with thickness configuration 1-2-1 ($Al/Al2O3$))

n	Results	Mesh	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
0	2 D FE	20 X 20	1.8291	4.3907	4.3907	6.1764	6.1764
	Present	12 X 12	1.8211	4.3407	4.3407	6.1788	6.1788
	Present	16 X 16	1.8226	4.3459	4.3459	6.1710	6.1710
	Present	20 X 20	1.8232	4.3484	4.3484	6.1675	6.1675
0.5	2 D FE	20 X 20	1.3945	3.3903	3.3903	5.2557	5.5925
	Present	12 X 12	1.3900	3.3551	3.3551	5.1817	5.5998
	Present	16 X 16	1.3911	3.3590	3.3590	5.1967	5.5927
	Present	20 X 20	1.3916	3.3610	3.3610	5.2039	5.5895
1	2 D FE	20 X 20	1.2818	3.1266	3.1266	4.8601	5.4507
	Present	12 X 12	1.2754	3.0885	3.0885	4.7835	5.4501
	Present	16 X 16	1.2764	3.0921	3.0921	4.7975	5.4433
	Present	20 X 20	1.2768	3.0939	3.0939	4.8040	5.4401

Non-dimensionalised frequencies in Tables 4 and 5 are obtained using following,

$$\bar{\omega} = \omega H \sqrt{\frac{\rho_m}{E_m}}$$

Where ρ_m is density of metal and E_m is Young's modulus of metal.

3.2.2. Results for Type 2 sandwich plate

The results of the non-dimensionalised natural frequencies for the all-round simply supported square FG sandwich plates with a/h = 10 and thickness configurations 1-2-1 and 2-1-1 are presented in Tables 6 and 7 respectively for various values of n. The results are compared with the 2 D finite element (FE) results obtained using the commercial software ANSYS [16]. ANSYS [16] results are obtained by dividing the top and the bottom FG layers of plate each into 25 layers in the thickness direction. And the central homogeneous core part is considered as one single layer. Thus a complete sandwich plate is split into 51 layers in the thickness direction. It is assumed that the material properties are different in the different layers but in a layer they are assumed to be constant and are obtained at the midpoint of each layer. The 2 D finite element (FE) ANSYS [16] results are obtained by using Shell 181 element. 20 X 20 mesh size is used to obtained ANSYS results. It appears that though the results are quite close to the results obtained by ANSYS, they are not converging to ANSYS results, as ANSYS results are based of FSDT. It is also observed that for this case also like Type 1 plate, as the value of volume fraction index n increases the non-dimensionalised frequencies decrease for all the thickness configurations. For n = 0 both the models give same results as the plate becomes isotropic for all the thickness configurations. But for other values of n the results of the non-dimensionalised frequencies obtained by using MT model are lower than that obtained by using ROM model for all the thickness configurations like that observed in single layer plate. It is also noticed that as n increases the difference between the results obtained using ROM model and that obtained using MT model reduces.

Table 7: First six non-dimensionalised frequencies for all round simply supported square Type 2 sandwich plate with thickness configuration 2-1-1 ($Al/Al2O3$))

n	Results	Mesh	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
0	2 D FE	20 X 20	1.8291	4.3907	4.3907	6.1764	6.1764
	Present	12 X 12	1.8211	4.3407	4.3407	6.1788	6.1788
	Present	16 X 16	1.8226	4.3459	4.3459	6.1710	6.1710
	Present	20 X 20	1.8232	4.3484	4.3484	6.1675	6.1675
0.5	2 D FE	20 X 20	1.2887	3.1659	3.1659	4.9497	5.2577
	Present	12 X 12	1.2805	3.0955	3.0955	4.7864	5.2611
	Present	16 X 16	1.2816	3.0989	3.0989	4.7999	5.2545
	Present	20 X 20	1.2820	3.1005	3.1005	4.8064	5.2515
1	2 D FE	20 X 20	1.1574	2.8500	2.8500	4.4651	4.9995
	Present	12 X 12	1.1435	2.7735	2.7735	4.3006	4.9950
	Present	16 X 16	1.1443	2.7764	2.7764	4.3124	4.9888
	Present	20 X 20	1.1448	2.7778	2.7778	4.3181	4.9860

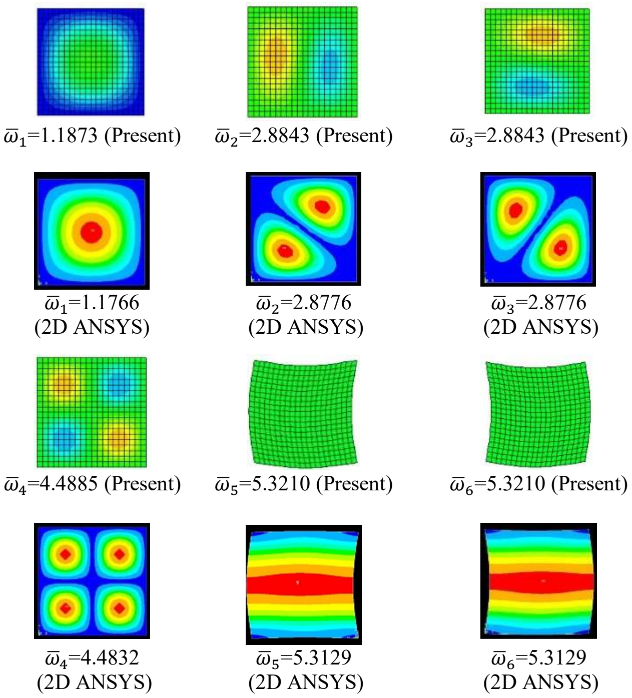

Fig.3. First six mode shapes of all round simply supported square Type 2 plate with thickness configuration 1-2-1 for volume fraction index n = 2 ($Al/Al2O3$)

The first six mode shapes for a sandwich all round simply supported square plate with thickness configuration 1-2-1 for volume fraction index n = 2 obtained using a mesh size of 20 X 20 are shown in (Fig. 3). The non-dimensionalised natural frequencies for all first six modes are matching with that of [16]. It is once again observed that the present formulation is capable of capturing the shear modes also.

4. Conclusions

The results for the non-dimensionalised natural frequencies obtained with the developed four node discrete Kirchhoff quadrilateral element, based on Reddy's third order theory, for free vibration response of FG sandwich Type 1 and Type 2 plates are compared in this work with 2 D analytical and finite element results available in literature. It is observed that for different thickness configuration, with two different scheme of material variation, the present element is exhibiting excellent performance for all the cases considered in this study. It is also observed that, the present formulation is suitable for capturing the modes, which the 2 D analytical solution fails to capture. It is also observed that the present results obtained with coarser mesh are also satisfactory.

Disclosures

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