

Model reduction technique for Bayesian model updating of structural parameters using simulated modal data

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Abstract

An attempt has been made to study the effectiveness of model reduction technique for Bayesian approach of model updating with incomplete modal data sets. The inverse problems in system identification require the solution of a family of plausible values of model parameters based on available data. Specifically, an iterative model reduction algorithm is proposed based on a non-linear optimization method to solve the transformation parameter such that no prior choices of response parameters are required. The modal ordinates synthesized at the unmeasured degrees of freedom (DOF) from the reduced order model are used for a better estimate of likelihood functions. The reduced-order model is subsequently implemented for updating of unknown structural parameters. The present study also synthesizes the mode shape ordinates at unmeasured DOF from the reduced order model. The efficiency of the proposed model reduction algorithm is further studied by adding noises of varying percentages to the measured modal data sets. The proposed methodology is illustrated numerically to update the stiffness parameters of an eight-story shear building model considering simulated datasets contaminated by Gaussian error as evidence. The capability of the proposed model reduction algorithm coupled with Markov Chain Monte Carlo (MCMC) algorithm is compared with the case where only MCMC algorithm is used to investigate their effectiveness in updating model parameters. The numerical study focuses on the effect of reduced number of measurements for various measurement configurations in estimating the variation of errors in determining the modal data. Subsequently, its effects in reducing the uncertainty of model updating parameters are investigated. The effectiveness of the proposed model reduction algorithm is tested for number of modes equal to the number of master DOFs and gradually decrease of mode numbers from the number of master DOFs.

Keywords: Model Updating, Bayesian approach, Markov Chain Monte Carlo, Model Reduction, Incomplete Data

1. Introduction

The global health monitoring of structures has been enhanced by remarkable development of contemporary experimental paradigms and computational approaches in the past decades. In this regard, the worldwide efforts of model updating over the years have motivated the increase of information extraction techniques to understand performance and behaviour of structures under real environmental conditions [1-4]. Model reduction in large finite element (FE) models is used significantly to reduce both computational burden and increased experimental investigations for updating model parameters in a probabilistic framework [5-7]. The model reduction technique basically evades the full model run by using dynamic response data thus avoiding significant re-analyses. It involves a wide range of applications not only in the fields of structural or earthquake engineering but also in structural health monitoring, material science, etc. The problem basically involves in obtaining a reduced-order model of structure through a correct transformation utilizing measured responses at only a limited number of locations [8]. It is also used to determine the unknown responses in

terms of the known responses i.e. at the measured DOFs. It enhances the attainment of dynamic responses accurately such that these responses are utilized in model updating practice significantly. Possibly Guyan's static reduction was the most popular and earliest method of model reduction [9]. It has been observed that static reduction never reproduces any of the natural frequencies of the original analytical model. The higher frequency modes are the least accurate since static reduction ignores the inertia terms which are critical to the accurate estimation of the higher natural frequencies [8]. The centre frequency in the range of interest is one choice although a geometric mean is used [8].

Various model reduction techniques such as Improved Reduction System (IRS), System Equivalent Reduction Expansion Process (SEREP), iterated IRS method (IIRS) were used to produce the transformation between the master and slave co-ordinates [8, 10]. It has been found that the problem with IIRS and SEREP methods are that the performance of the reduced model requires proper choice of master DOFs otherwise it usually takes a huge computational time [8]. In case of limited measurements, the

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real test of the improved model reduction approach is whether the dynamic behaviour of structure of interest are significant in probabilistic model updating of model parameters and for comprehensive quantification of their uncertainties [11]. To alleviate the computational demands for performing dynamic eigenvalue analysis of full model, a model reduction technique for non-linear systems is considered in the present formulation. The model reduction method proposed in the present study can be implemented on methods of component mode synthesis or methods of substructure coupling. The transformation matrix in the model reduction is used to form a reduced order model of the entire structure which is subsequently used in performing the dynamic eigenvalue analysis for model parameter updating. The inverse problem in structural system identification requires the solution of a family of plausible values of model parameters rather than the search of single optimal parameter vector [12-14]. In this regard, Bayesian probability quantifies relative plausibility of current state of each model parameters with respect to the previous state by stochastically estimating likelihoods of the difference of the experimentally observed data and predicted data [15-17]. The Markov Chain Monte Carlo (MCMC) simulation technique has significantly enhanced classical Bayesian approach by eliminating the need to evaluate the normalizing constant and enabling the chain to converge to the globally optimum region after propagation through the parameter space in lesser computational time and found to be applicable for higher parameter uncertainty [18-20]. The computational cost of basic MCMC technique is relatively less when compared to other stochastic simulation methods as it samples from arbitrary prior pdf and stochastically generates the chain based on the relative plausibility of the current state of the model parameters with respect to the previous state of the proposal distribution using Metropolis Hastings (MH) algorithm. In this regard, the re-analyses of full models can be suppressed by introducing model-reduction techniques such as Guyan's reduction, substructure coupling [6-7]. The MCMC based model updating technique necessitates drawing a huge number of samples for populating the important region in the uncertain parameter space. A dynamic eigenvalue analysis of the model is required at each step for the structural parameter samples generated by MCMC. This is due to the fact that the evaluation of likelihood function involves the requirement for the calculation of predicted evidence. The model reduction algorithm proposed here is used to obtain a reduced order model with better estimation of transformation parameters than IIRS based approach besides overcoming its drawbacks.

The present study attempts to develop the model reduction technique based on an iterative non-linear least-square method in the form of Gauss-Newton (GN) approach utilizing measured responses at limited locations. The present study basically obtains the transformation inherent in the model reduction technique and synthesizes the responses at the unmeasured DOFs. The numbers of measured locations are gradually reduced and the choice of the measured DOFs is made arbitrary to test the effectiveness of the proposed approach. This study is performed considering both the cases when the number of modes is equal to the number of measured DOF and when

the number of modes is less than the number of measured DOFs. The effectiveness of the proposed approach are also tested by adding noises to the measured data sets at various locations. The transformation deriving the reduced-order model using the proposed model reduction approach is applied to a Bayesian based MCMC framework for updating of model parameters and its posterior estimation. The proposed techniques are illustrated numerically by considering an eight-storey shear building model to study the accuracy of estimating model parameters and associated uncertainties considering incomplete modal data sets. Further, to study the effectiveness of the proposed approach, a comparative study with regard to accuracy of the proposed approach is made with the IIRS based approach of model reduction.

2. Model Reduction

The model reduction technique basically evades the full model run for each iteration using dynamic response data. The master DOFs corresponds to the measured ones and the slave DOFs corresponds to those of the unmeasured ones. The formulation of a generalized eigenvalue problem of n -DOF linear system with the mass and stiffness matrices governed by master (m) and slave (s) DOFs, expressed as:

$$\mathbf{K}(\boldsymbol{\theta}) \begin{Bmatrix} \boldsymbol{\Phi}_{mm} \\ \boldsymbol{\Phi}_{sm} \end{Bmatrix} = \mathbf{M} \begin{Bmatrix} \boldsymbol{\Phi}_{mm} \\ \boldsymbol{\Phi}_{sm} \end{Bmatrix} \boldsymbol{\Lambda}_{mm} \quad (1)$$

where, n is the total number of DOFs of a structure, $\boldsymbol{\theta} = \{\theta_1, \theta_2, \dots, \theta_{N_p}\} \in \mathbf{R}^{N_p}$ is the structural model parameter vector (where N_p is the number of model parameters of the structure) associated with each DOF of the structure, $\boldsymbol{\Phi} \in \mathbf{R}^{n \times m}$ is the normalized mode shape matrix considering first m modes with m equal to n , $\boldsymbol{\Phi} = \{\boldsymbol{\Phi}_{mm} \quad \boldsymbol{\Phi}_{sm}\}^T$, $\boldsymbol{\Lambda}_{mm} \in \mathbf{R}^{m \times m}$ is the diagonal eigenvalue matrix consisting of squares of modal frequencies of structure ω^2 for first m modes [7]. The partitioning of the mass and stiffness matrices governed by master and slave DOFs with their numbers denoted by m and s respectively, expressed as:

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{sm} & \mathbf{M}_{ss} \end{bmatrix}, \quad \mathbf{K}(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{K}_{mm}(\boldsymbol{\theta}) & \mathbf{K}_{ms}(\boldsymbol{\theta}) \\ \mathbf{K}_{sm}(\boldsymbol{\theta}) & \mathbf{K}_{ss}(\boldsymbol{\theta}) \end{bmatrix} \quad (2)$$

where, $\mathbf{M}_{mm}$ and $\mathbf{K}_{mm}(\boldsymbol{\theta})$, $\mathbf{M}_{ss}$ and $\mathbf{K}_{ss}(\boldsymbol{\theta})$ represents the elements of $\mathbf{M}$ and $\mathbf{K}(\boldsymbol{\theta})$ matrices associated only with m and s respectively; $\mathbf{M}_{ms}$ and $\mathbf{K}_{ms}(\boldsymbol{\theta})$ represents the elements of $\mathbf{M}$ and $\mathbf{K}(\boldsymbol{\theta})$ matrices associated with both m and s ; $\mathbf{M}_{sm}$ and $\mathbf{K}_{sm}(\boldsymbol{\theta})$ are basically similar as $\mathbf{M}_{ms}$ and $\mathbf{K}_{ms}(\boldsymbol{\theta})$ respectively only with the off-diagonals positions getting transposed. Similarly, $\boldsymbol{\Phi}_{mm}$ of the mode shape matrix $\boldsymbol{\Phi}$ contains the mode shape ordinates corresponding to m for first m number of modes and $\boldsymbol{\Phi}_{sm}$ of the mode shape matrix $\boldsymbol{\Phi}$ contains the mode shape ordinates corresponding to s for first m number of modes. The mode shape ordinates corresponding to slave DOFs s is represented in terms of master DOFs m for first m number of modes as,

$$\boldsymbol{\Phi}_{sm} = \mathbf{t} \boldsymbol{\Phi}_{mm} \quad (3)$$

where, $\mathbf{t} \in \mathbf{R}^{s \times m}$ is a transformation matrix [7]. Eq. (3) can be substituted after combining Eq. (1) and Eq. (2) into the second set of the resulting equation to obtain the following expression as [7],

$$\mathbf{t} = -\mathbf{K}_{ss}^{-1}(\boldsymbol{\theta})\mathbf{K}_{ms}^T(\boldsymbol{\theta}) + \mathbf{K}_{ss}^{-1}(\boldsymbol{\theta})[\mathbf{M}_{ms}^T + \mathbf{M}_{ss}\mathbf{t}]\boldsymbol{\Phi}_{mm}\boldsymbol{\Lambda}_{mm}\boldsymbol{\Phi}_{mm}^{-1} \quad (4)$$

For the necessity of reducing the equations of motion to a reduced set of DOFs, the transformation operator is formed as,

$$\mathbf{T} = \{\mathbf{I} \quad \mathbf{t}\}^T \quad (5)$$

where, $\mathbf{I} \in \mathbf{R}^{m \times m}$ [7,9]. The mass and stiffness matrices of the reduced order model are obtained as,

$$\mathbf{M}_R = \mathbf{T}^T \mathbf{M} \mathbf{T} \quad \text{and} \quad \mathbf{K}_R(\boldsymbol{\theta}) = \mathbf{T}^T \mathbf{K}(\boldsymbol{\theta}) \mathbf{T} \quad (6)$$

where, $\mathbf{M}_R$ and $\mathbf{K}_R(\boldsymbol{\theta})$ are the mass and stiffness matrices of the reduced order model respectively [7].

It can be observed from the above formulations that the real challenge of obtaining a correct reduced order model is the proper estimation of the transformation matrix $\mathbf{t}$. This leads to the solution of transformation operator $\mathbf{T}$ and correspondingly the reduced-order model is solved. Various notable works has solved $\mathbf{t}$ through iterative processes which are briefly highlighted as:

2.1 Iterated Improved Reduced System (IIRS) Method

O'Callahan (1989) introduced a technique known as the IRS that is an improvement on the static reduction method. This in fact provided a perturbation to the transformation from the static case by including the inertia terms as pseudo static forces which is expressed as [8],

$$\mathbf{T}_{IRS} = \mathbf{T}_s + \mathbf{S} \mathbf{M} \mathbf{T}_s \mathbf{M}_R^{-1} \mathbf{K}_R \quad (7)$$

where, $\mathbf{S} = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{K}_{ss}^{-1} \end{bmatrix}$, $\mathbf{T}_s$ is the unknown static transformation operator containing the unknown parameter $\mathbf{t}$, $\mathbf{T}_{IRS}$ is the updated IRS based transformation operator and the rest have usual notations as defined [8]. Similarly the dynamic IRS (DIRS) transformation method which is expressed as [8],

$$\mathbf{T}_{DIRS} = \mathbf{T}_d + \mathbf{S}_d \mathbf{M} \mathbf{T}_d \mathbf{M}_R^{-1} \mathbf{D}_R \quad (8)$$

where, $\mathbf{S}_d = \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{D}_{ss}^{-1} \end{bmatrix}$, $\mathbf{D}_{ss} = \mathbf{K}_{ss} - \Omega^2 \mathbf{M}_{ss}$,

$\mathbf{D}_R = \mathbf{K}_R - \Omega^2 \mathbf{M}_R$, Ω is the given frequency, $\mathbf{T}_d$ is the unknown dynamic transformation operator containing the unknown parameter $\mathbf{t}$, $\mathbf{T}_{DIRS}$ is the updated DIRS based transformation operator and the rest have usual notations as defined [8]. The transformation for the dynamic reduction method was found to be correct at the given frequency Ω . Subsequently, an iterative improved reduced system (IIRS) method was implemented to obtain an improved estimate of the IRS transformation matrix for both static and dynamic methods by [8]. The iterations for static method is shown as [8],

$$\mathbf{T}_{IRS,i+1} = \mathbf{T}_s + \mathbf{S} \mathbf{M} \mathbf{T}_{IRS,i} \mathbf{M}_R^{-1} \mathbf{K}_{IRS,i} \quad (9)$$

where, i denotes the i th iteration and $\mathbf{T}_{IRS,i+1}$ indicates the transformation operator for IIRS method [8]. The subsequent iterations for dynamic method is shown as [8],

$$\mathbf{T}_{IRS,i+1} = \mathbf{T}_s + \mathbf{S} \mathbf{M} \mathbf{T}_{IRS,i} \mathbf{M}_R^{-1} \mathbf{K}_{IRS,i} \quad (10)$$

where, i denotes the i th iteration [8]. The transformation operator for IIRS method is represented as [8],

$$\mathbf{T}_{IRS} = \begin{bmatrix} \mathbf{I} \\ \mathbf{t}_{IRS} \end{bmatrix} \quad (11)$$

where,

$$\mathbf{t}_{IRS} = -\mathbf{K}_{ss}^{-1} \mathbf{K}_{sm} + \mathbf{K}_{ss}^{-1} (\mathbf{M}_{sm} + \mathbf{M}_{ss} \mathbf{t}_{IRS}) \mathbf{M}_{IRS}^{-1} \mathbf{K}_{IRS} \quad (12)$$

where, $\mathbf{t}_{IRS}$ is the transformation matrix for the iterated IRS method exactly similar to the transformation matrix $\mathbf{t}$ in Eq. (4) [8]. Consider λ as eigenvalues of the full system with eigenvector $\boldsymbol{\Phi}_m$ for the reduced system. The IRS based mass ($\mathbf{M}_{IRS}$) and stiffness ($\mathbf{K}_{IRS}$) matrices are obtained as [8],

$$\mathbf{M}_{IRS}^{-1} \mathbf{K}_{IRS} \boldsymbol{\Phi}_m = \lambda \boldsymbol{\Phi}_m \quad (13)$$

Eq. (12) and Eq. (13) are iterated to obtain the reduced-order model of the entire system. It has also been observed from the previous works that the number of iterations to obtain convergence of transformation matrix in IIRS method was found to depend on the properties of the full system and the choice of master DOFs. Besides, there is an occurrence of spurious natural frequencies in the reduced-order model which was found to vary from one step to another step even if there was 2% variation in the choice of initial frequency. Along with this, the performance of the model reduction based on the consideration of the number of modes less than m is not performed. It is also closely related to the fact that if m number of DOFs are measured then m number of modes are to be considered for model reduction problem which becomes computationally inefficient.

To circumvent the aforementioned drawbacks, the present study attempts to develop a reduced-order model based on an iterative least-square optimization algorithm to determine the unknown parameter $\mathbf{t}$ from Eq. (4) iteratively. The present study implements GN process and develops the model reduction algorithm accordingly to evaluate transformation matrix $\mathbf{t}$ from Eq. (4) which is unknown iteratively to obtain an efficient reduced-order model. The proposed technique focuses also on the performance of the reduced system based on the consideration that the number of modes considered is less than m number of DOFs. To achieve the criterion, the number of modes considered for model reduction is gradually reduced from the number m to observe the change in the accuracy of predicted responses from the reduced-order model. The improved formulation for the above criterion is briefly stated as:

Let m_r be the number of modes considered such that $m_r < m$. The mode shape matrix is represented as $\boldsymbol{\Phi} = \{\boldsymbol{\Phi}_{mm_r} \quad \boldsymbol{\Phi}_{sm_r}\}^T$. Eq. (1) is modified accordingly. The mode shape ordinates corresponding to slave DOFs s is represented in terms of master DOFs m for m_r number of modes as,

$$\boldsymbol{\Phi}_{sm_r} = \mathbf{t} \boldsymbol{\Phi}_{mm_r} \quad (14)$$

where, $\mathbf{t} \in \mathbf{R}^{s \times m_r}$. Eq. (13) can be substituted after combining Eq. (1) after modification as discussed previously and Eq. (2) into the second set of the resulting equation to obtain the following expression as,

$$\mathbf{t} = -\mathbf{K}_{ss}^{-1}(\boldsymbol{\theta})\mathbf{K}_{ms}^T(\boldsymbol{\theta}) + \mathbf{K}_{ss}^{-1}(\boldsymbol{\theta})[\mathbf{M}_{ms}^T + \mathbf{M}_{ss}\mathbf{t}]\boldsymbol{\Phi}_{mm_r}\boldsymbol{\Lambda}_{mm_r}\boldsymbol{\Phi}_{mm_r}^{-1} \quad (15)$$

Eq. (14) is solved and transformation parameter $\mathbf{t}$ is obtained using the proposed GN based model reduction algorithm. $\boldsymbol{\Phi}_{mm_r}^{-1}$ is solved using pseudo-inverse operator. To obtain the

dynamic responses of the reduced-order model, the dynamic eigenvalue problem is solved as,

$$[\mathbf{K}_R(\boldsymbol{\theta}) - \omega_{m_r}^2 \mathbf{M}_R] \boldsymbol{\Phi}_{mm_r} \quad (16)$$

3. Proposed Gauss-Newton based Model Reduction algorithm

The present study attempts to implement the Gauss-Newton (GN) based least squares optimization to obtain a simplified iterative solution to the model reduction approach of dynamic responses. A function $f(\mathbf{x})$ minimized as a sum of squares in the least squares problem can be represented as,

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) = \frac{1}{2} \|\mathbf{F}(\mathbf{x})\|_2^2 = \frac{1}{2} \sum_i F_i(\mathbf{x})^2 \quad (17)$$

where, $\|\cdot\|_2$ denotes the Euclidean norm of the least square function $F(\mathbf{x})$. It is also found that while obtaining realistic target trajectories the residual of $\|\mathbf{F}(\mathbf{x})\|$ for such problems is usually small. The procedure of calculating the sensitivities is expensive for a large finite element model. Hence, we have introduced least squares solutions for $(x_{i+1} - x_i)$ in the form of $\|\mathbf{F}(\mathbf{x})\|$. The residual is obtained until the difference is sufficiently small for convergence to be deemed to have occurred. It is important to note that the GN approach depends upon the availability of an initial estimate of the parameter to be determined. The quality of the initial estimate can affect both the speed of convergence and whether or not convergence to the 'true' parameters is achieved. The parameter estimation problem is, of course, often posed as a problem of constrained minimisation and in the case of non-linearity in parameters, a particular minimum is sought on a surface which contains many minima and maxima. Usually either a local minimum is sought, when there is confidence in the initial model, or else the problem is to determine the unique global minimum. In the present study, the Gauss-Newton algorithm (GNA) basically obtains a search direction at each of i -th iteration of the optimization problem in the form of a least squares solution. The resulting functional relation is expressed as,

$$\mathbf{Q}(\mathbf{x}) = \sum_i F_i(\mathbf{x}) \cdot \mathbf{H}_i(\mathbf{x}) \quad (18)$$

where, $\mathbf{H}(\mathbf{x})$ is the Hessian matrix of the least squares function. The GN formulation is expressed as,

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{J}(\mathbf{x}_i) d_i - \mathbf{F}(\mathbf{x}_i)\|_2^2 \quad (19)$$

where, $\mathbf{J}(\mathbf{x}_i)$ is the Jacobian matrix of the least squares function. The GNA establishes line search procedures with a quasi-Newton direction when the higher order terms of $\mathbf{Q}(\mathbf{x})$ in Eq. (17) are ignored. The search direction ensures that the function $f(\mathbf{x})$ decreases at each iteration. In the present study, $\mathbf{f}$ is the relation that determines the unknown parameter i.e., $\mathbf{t}_i \in \mathbb{R}^{s \times m}$ at the i -th iteration is obtained from Eq. (4) or Eq. (15) as,

$$\mathbf{f}(\mathbf{t}_i) = \mathbf{t}_i - (-\mathbf{K}_{ss}^{-1}(\boldsymbol{\theta}) \mathbf{K}_{ms}^T(\boldsymbol{\theta}) + \mathbf{K}_{ss}^{-1}(\boldsymbol{\theta}) [\mathbf{M}_{ms}^T + \mathbf{M}_{ss} \mathbf{t}_i] \boldsymbol{\Phi}_{mm} \boldsymbol{\Lambda}_{mm} \boldsymbol{\Phi}_{mm}^{-1}) \quad (20)$$

The GN algorithm performs Hessian update under the following conditions:

- The Hessian has always maintained to be positive definite so that the direction of search is always in a descent direction.
- The line-search step length parameter for every iteration is tuned in such a way that it is compensated for the case when the approximation to the Hessian is monotonically increasing or decreasing.
- The residual $\|\mathbf{F}(\mathbf{x})\|$ drops below a certain threshold.
- A maximum number of iterations is completed.

This technique is particularly effective in solving systems of first order non-linear equations that have both large sets of non-linear equations and large parameter sets. In GNA method, the conditioning of the equations is avoided by using the QR decomposition of $\mathbf{J}(\mathbf{x}_i)$ and applying the decomposition to $\mathbf{F}(\mathbf{x}_i)$. This evades the unnecessary occurrence of errors besides keeping the residual $\|\mathbf{F}(\mathbf{x})\|$ minimum. The unknown parameter $\mathbf{t}$ is thus developed for the implementation of the model reduction algorithm for the updating of model parameters.

4. Bayesian approach of Model Updating

In the Bayesian approach, subjective judgements based on intuition, experience or indirect information is systematically incorporated with the observed data to obtain a balanced estimate of model parameters. The fundamental rule that governs Bayesian model updating is Bayes theorem [18] which states that given the observed data, the posterior probability function of the model depends on the likelihood function, the prior distribution and the evidence. This can be mathematically expression as,

$$p(\boldsymbol{\theta} | \mathbf{D}) = \frac{p(\mathbf{D} | \boldsymbol{\theta}) p(\boldsymbol{\theta})}{p(\mathbf{D})} \quad (21)$$

where, $\boldsymbol{\theta}$ is the vector of updating parameters, $\mathbf{D}$ is the available data, $P(\mathbf{D} | \boldsymbol{\theta})$ is the likelihood that represents the probability of the data $\mathbf{D}$ when a belief of $\boldsymbol{\theta}$ is taken as true, $P(\boldsymbol{\theta})$ is the prior pdf that contains the knowledge about $\boldsymbol{\theta}$ before observing the data, $P(\boldsymbol{\theta} | \mathbf{D})$ is the posterior pdf and $P(\mathbf{D})$ is the normalization factor (the evidence given by the data). The difference between the measured responses and the predicted outputs of evidences from a model is the prediction error which can be defined as,

$$e(\boldsymbol{\theta}) = Y - X(\boldsymbol{\theta}) \quad (22)$$

where, $e(\boldsymbol{\theta})$ is the prediction error, Y is the measured response and $X(\boldsymbol{\theta})$ is the predicted output of the evidences from the model. The likelihood function of evidences for the outcome (Y) can be written as:

$$p(Y | \boldsymbol{\theta}, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{[e(\boldsymbol{\theta})]^2}{2\sigma^2} \right\} = \frac{1}{\sigma \sqrt{2\pi}} \exp \left\{ -\frac{[Y - X(\boldsymbol{\theta})]^2}{2\sigma^2} \right\} \quad (23)$$

The evidences ($\mathbf{D}$) of the model parameters in the present study are the modal data of a structure consisting of

modal frequencies. In Bayesian inference, the prior distributions of parameters describing a model is necessary. The prior distribution of the parameters θ i.e. $P(\theta)$ is assumed to be exponential and the different components of θ is assumed to be independent. The posterior distribution of the updating parameters can be now obtained by Bayes theorem using Eq. (1). It may be noted that apart from model parameters (θ), the updating parameters vector involve the variances of the frequency. Thus, the posterior values of the parameters in Eq (1) is redefined as, $\bar{\theta} = \left\{ \theta^T, \left\{ \sigma_{\omega_i}^2 \right\}_{i=1}^m \right\}^T$. In the present study, the MCMC technique is used to obtain samples from the posterior distribution.

5. Bayesian MCMC based proposed Model Reduction Approach

The MCMC allows efficient generation of random samples as a sequence of a “Markov Chain” according to an arbitrarily given probability distribution. The MH based MCMC approach is used. The proposal distribution, f governs the distribution of the candidate and affects the transition of the Markov chain from one state to another. In the present study, Gaussian distribution is used as the proposal distribution to draw samples from the high dimensional posterior distribution. The acceptance criteria ratio of MH algorithm in the following manner,

$$r = \left\{ \frac{p(\omega_i | \bar{\theta}, \sigma_{\omega_i}^2) p(\sigma_{\omega_i}^2 | \bar{\theta}) p(\bar{\theta})}{p(\omega_i | \theta, \sigma_{\omega_i}^2) p(\sigma_{\omega_i}^2 | \theta) p(\theta)} \right\} \quad (24)$$

The posterior parameter value $\bar{\theta}$ is then accepted either with probability α_a or rejected with a probability of $1 - \alpha_a$. The initial acceptance ratio is kept low to avoid any bias of updating the parameter value right from the start of the algorithm. The measured dynamic response data in the form of natural frequencies and mode shapes are implied in Eq. (4) along with the stiffness matrix $K(\theta)$ partitioned in Eq. (2) based on the prior value of model parameter θ . Then GN algorithm evaluates t by solving Eq. (4) which is used to synthesize the responses at the slave DOFs from the measured responses at master DOFs as in Eq. (3). The reduced-order model (M_R, K_R) is evaluated using Eq. (5) and Eq. (6) subsequently to obtain the predicted responses which are synthesized for the slave DOFs using t . This forms for a complete mode matching between the actual and predicted responses. This performs one set of subroutine for model reduction in the same iteration of MCMC. The candidate sample for model parameter θ obtained by performing random walk in the main MCMC simulating algorithm is again passed through model reduction subroutine to evaluate another set of t for complete mode matching in the same MCMC iteration. This completes a full set of model reduction process for a single iteration of

MCMC. The synthesized predicted and actual responses for prior and candidate samples are used to obtain their likelihoods and then incorporated in the Bayesian decision making statement. This process is repeated for each iteration of the MCMC based model updating algorithm. The computational requirement for implementing the model reduction subroutine is higher only for the first iteration of MCMC algorithm as two sets of it needs to be performed. But for every other iterations, only one set of model reduction needs to be performed corresponding to the candidate sample obtained from the proposal distribution. This is due to the fact that since the transformation matrix t is evaluated for the prior and the candidate sample for the preceding iteration, hence only the candidate sample for the next iteration needs to be performed. The MH based acceptance criteria is updated based on the natural frequencies and mode shapes obtained from the reduced-order model. The error variance in the Bayesian based parameter updating is considered for the natural frequencies only.

6. Numerical Study

An eight-storied shear building model as shown in Figure 1(a) to (e) is taken up for numerical elucidation of the proposed Bayesian model updating technique. The building has uniform nominal stiffness, $k_i (2 \times 10^8 \text{ N/mm})$ and mass $m_i (1 \times 10^5 \text{ kg})$ associated with each storey. An updating parameter θ_i (with $i=1, 2, \dots, 8$) is enforced on each of these stiffness parameters. The multiplication of these two factors represents the actual stiffness of the i -th storey i.e. ($K_i = k_i \times \theta_i$). The unknown parameters θ_i are obtained under Bayesian inference. To apply the algorithm, synthetic modal response data are used in the present study. To simulate the evidence, the modal data of the structure are evaluated for a set of chosen values of the unknown updating parameter, θ_i . The frequencies and mode shapes of the first two modes are considered as the mean data (ω_m, ϕ_m). The modal responses are artificially simulated from a Gaussian distribution with the estimated modal data as the mean value and 10%, 15% and 20% coefficient of variation. Following conditions of model updating are performed with incomplete measurements:

Case 1: The first case considers that the six measurements of modal data are available as shown in Fig. 1(a).

Case 2: The first case considers that the five measurements of modal data are available as shown in Fig. 1(b).

Case 3: The first case considers that the four measurements of modal data are available as shown in Fig. 1(c).

Case 4: The first case considers that the three measurements of modal data are available as shown in Fig. 1(d).

Case 5: The first case considers that the two measurements of modal data are available as shown in Fig. 1(e).

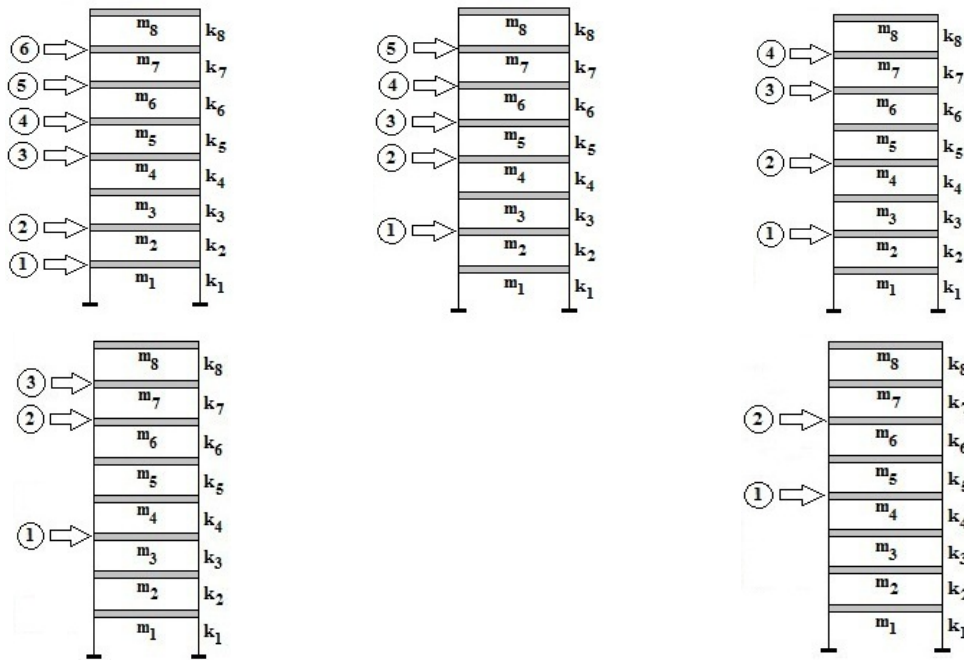


Fig. 1. Schematic diagrams of the eight-storied shear building showing the location of measured DOFs i.e. (a) at 6 locations, (b) at 5 locations, (c) at 4 locations, (d) at 3 locations and (e) at 2 locations.

The percentage of errors in obtaining natural frequencies and mode shapes for first three modes using LMA based model reduction with gradual reduction in N_m with respect to N_o and different noise percentages are summarized from Table 1 to 4. Table 1 to 2 gives a comparison of the accuracy in estimation of GNA and IIRS based model reduction for first and second modal frequencies. Table 3 to 4 gives a comparison of the accuracy

in estimation of GNA and IIRS based model reduction for first and second mode shapes. It can be observed that the proposed model reduction algorithm performs quite well in comparison to the traditional IIRS based model reduction technique irrespective of an arbitrary choice of master DOFs, presence of high noises in measured data and reduced number of measured modes.

Table-1. First modal frequency of structure using proposed model reduction approach.

Actual frequency (rad/s)	No. of measured modes	Noise (%)	The percentage of errors of the first natural frequency									
			GNA based Model reduction					IIRS based Model reduction				
			No. of measured DOFs					No. of measured DOFs				
			6	5	4	3	2	6	5	4	3	2
First mode = 175.69	6	10	1.72	-	-	-	-	1.95	-	-	-	-
		15	2.25	-	-	-	-	2.76	-	-	-	-
		20	3.00	-	-	-	-	4.02	-	-	-	-
	5	10	2.74	2.12	-	-	-	2.76	2.41	-	-	-
		15	3.26	2.75	-	-	-	3.68	3.33	-	-	-
		20	3.77	3.58	-	-	-	5.40	6.32	-	-	-
	4	10	4.25	3.25	3.15	-	-	4.48	5.06	3.56	-	-
		15	4.50	4.59	5.25	-	-	5.63	6.55	4.94	-	-
		20	5.25	6.51	7.7	-	-	8.16	9.66	8.74	-	-
	3	10	5.75	4.75	4.9	4.2	-	6.9	7.93	9.27	4.67	-
		15	6.52	5.75	6.3	5.6	-	8.51	10.09	11.51	6.55	-
		20	7.56	7.25	8.4	7.35	-	10.46	12.6	14.95	11.54	-
	2	10	6.25	6.25	7.35	5.95	4.9	9.89	11.04	12.65	9.66	7.82
		15	7.25	7.75	9.1	8.05	6.65	11.27	12.65	14.95	13.8	10.05
		20	7.78	9.01	11.55	10.15	8.05	12.65	14.95	17.25	16.18	13.84

Table-2. Second modal frequency of structure using proposed model reduction approach.

Actual frequency (rad/s)	No. of measured modes	Noise (%)	The percentage of errors of the second natural frequency									
			GNA based Model reduction					IIRS based Model reduction				
			No. of measured DOFs					No. of measured DOFs				
			6	5	4	3	2	6	5	4	3	2
Second mode = 628.42	6	10	1.65	-	-	-	-	2.41	-	-	-	-
		15	2.41	-	-	-	-	3.22	-	-	-	-
		20	2.73	-	-	-	-	4.37	-	-	-	-
	5	10	2.42	2.72	-	-	-	3.10	2.99	-	-	-
		15	2.70	3.34	-	-	-	4.14	3.79	-	-	-
		20	3.15	4.27	-	-	-	5.86	6.78	-	-	-
	4	10	3.31	3.45	3.61	-	-	4.94	5.52	4.02	-	-
		15	3.60	3.75	4.75	-	-	6.21	7.01	6.55	-	-
		20	4.24	4.35	6.08	-	-	8.51	10.12	9.31	-	-
	3	10	4.20	4.29	4.56	4.62	-	7.13	8.39	9.78	5.17	-
		15	4.65	4.95	5.51	5.52	-	8.97	10.46	11.5	7.93	-
		20	5.72	5.85	7.22	7.04	-	10.69	12.65	14.95	9.89	-
	2	10	4.51	4.95	5.70	6.16	5.28	10.23	11.38	12.65	10.23	8.39
		15	5.14	5.78	6.84	7.73	6.16	11.58	12.65	14.95	12.65	10.46
		20	5.55	6.67	8.17	9.24	7.48	12.65	14.95	17.25	16.19	13.88

Table-3. First mode shape of structure using proposed model reduction approach.

Actual frequency (rad/s)	No. of measured modes	Noise (%)	The percentage of errors of the first mode shape									
			GNA based Model reduction					IIRS based Model reduction				
			No. of measured DOFs					No. of measured DOFs				
			6	5	4	3	2	6	5	4	3	2
First mode = {0.18, 0.30, 0.70, 0.78, 0.88, 0.92, 0.96, 1}	6	10	1.34	-	-	-	-	2.07	-	-	-	-
		15	1.68	-	-	-	-	2.99	-	-	-	-
		20	2.18	-	-	-	-	4.37	-	-	-	-
	5	10	2.01	1.71	-	-	-	2.99	2.64	-	-	-
		15	2.35	2.37	-	-	-	4.02	3.56	-	-	-
		20	2.68	2.98	-	-	-	5.86	5.29	-	-	-
	4	10	3.02	2.92	2.68	-	-	4.83	5.52	3.91	-	-
		15	3.19	4.15	4.42	-	-	6.09	7.13	5.17	-	-
		20	3.86	5.34	6.52	-	-	8.85	9.89	7.13	-	-
	3	10	4.22	4.35	4.16	4.03	-	7.46	8.62	10.45	4.94	-
		15	4.71	5.14	5.98	5.58	-	9.22	10.46	11.59	6.78	-
		20	6.04	6.73	8.84	7.13	-	11.27	12.65	13.89	11.15	-
	2	10	4.53	5.34	5.72	5.89	5.17	9.54	10.81	12.65	10.46	9.66
		15	5.20	6.53	7.28	7.75	6.98	11.54	12.65	13.81	11.57	11.15
		20	6.55	8.11	8.84	9.92	8.97	12.65	14.95	16.19	14.95	12.58

Table-4. Second mode shape of structure using proposed model reduction approach.

Actual frequency (rad/s)	No. of measured modes	Noise (%)	The percentage of errors of the second mode shape									
			GNA based Model reduction					IIRS based Model reduction				
			No. of measured DOFs					No. of measured DOFs				
			6	5	4	3	2	6	5	4	3	2
Second mode = $\{-0.55, -0.78, -0.84, -0.66, -0.19, 0.13, 0.51, 1\}$	6	10	1.39	-	-	-	-	2.53	-	-	-	-
		15	1.97	-	-	-	-	3.33	-	-	-	-
		20	2.24	-	-	-	-	4.71	-	-	-	-
	5	10	1.85	2.20	-	-	-	3.345	3.22	-	-	-
		15	2.32	2.78	-	-	-	4.37	4.25	-	-	-
		20	2.78	3.48	-	-	-	6.32	5.75	-	-	-
	4	10	2.55	2.9	2.72	-	-	5.29	5.86	4.25	-	-
		15	3.24	3.13	3.67	-	-	6.67	7.47	5.63	-	-
		20	3.82	4.17	4.76	-	-	9.08	10.81	7.36	-	-
	3	10	3.48	3.71	3.53	2.99	-	7.59	8.97	10.35	5.52	-
		15	4.06	4.29	4.62	3.67	-	9.54	11.15	12.65	7.13	-
		20	4.98	5.22	5.57	4.35	-	11.38	13.8	14.95	9.08	-
	2	10	3.71	4.06	4.48	3.53	4.83	10.92	11.5	12.65	10.92	8.97
		15	4.17	4.75	5.30	4.35	5.58	11.5	12.65	14.95	12.65	10.81
		20	4.75	5.68	5.98	5.44	6.51	13.8	16.1	17.25	16.1	14.95

It can be noted from the tables that the proposed GNA based model reduction performs better than the IIRS based model reduction as the accuracy for GNA is significantly high compared to IIRS at all levels.

Now the effect of considering the proposed GNA based model reduction technique in the accuracy and associated uncertainties of estimated model parameters are studied. For this, the parameter identification is performed by utilizing MCMC algorithm in Bayesian framework and applying it in both model reduction methods. The likelihood function is developed as in Eq. (23) by constructing the reduced-order model from the proposed model reduction approach as well as the IIRS based approach. The prior pdf is assumed uniform to avoid any bias of choosing the parameter value from the bounded interval. For all the cases, the posterior samples are derived by MCMC algorithm and 10000 samples are implemented. For brevity, the posterior mean and standard deviation (std. dev.) of the predicted parameters at 20% noise level and considering least number of modes N_m are shown in Table 5. It can be readily noted from the table that the improved estimated parameters are

achieved when the proposed model reduction algorithm is considered. Also, the std. dev. values of model parameters are noted to be less for the proposed GN approach. It is also observed that obtaining a good estimate of modal responses is highly dependent on its subsequent estimation of model parameters as evident from its posterior statistics. In this regard, the proposed model reduction approach estimates the posterior distribution of model parameters quite well over the conventional IIRS based model reduction approach.

The posterior mean values of the parameters for both the cases are compared in Fig. 2. The plots correspond to mean values for the maximum noise of 20% noise with least number of modes i.e., N_m measured for each 'condition' i.e., number of N_o DOFs. It can be noted from the figures that when the proposed approach is considered, the estimated values are better than that of obtained by considering the IIRS based model reduction approach. The accuracy levels of estimating posterior values of parameters by the proposed approach with high noise percentage and least measured modes are almost similar. Thus, overall the GNA based approach is expected to be an efficient choice.

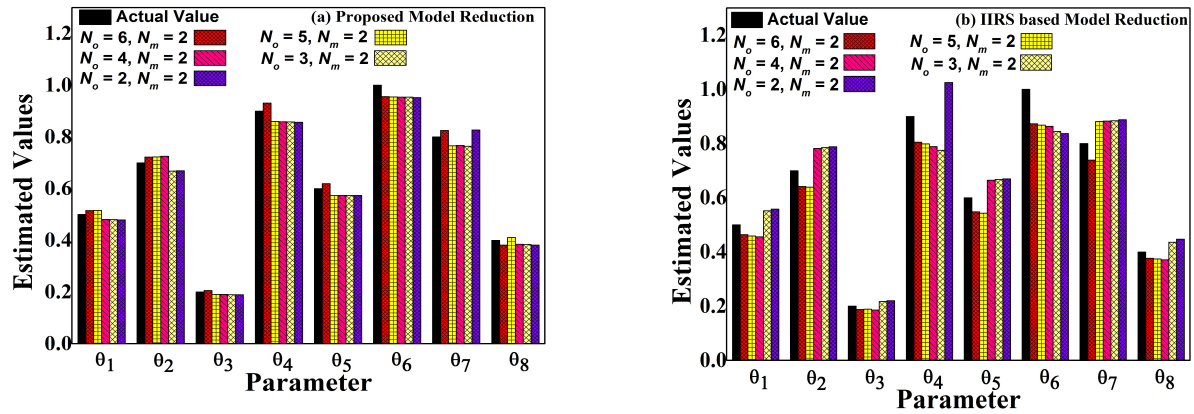


Fig. 2. Comparison of the posterior mean values and the actual values of the model parameters for 20% noise.

Table-5. Comparison of actual and predicted parameters with 20% noise and least number of modes.

Parameter (Actual values)	Method	Posterior mean and std. dev. (%) of parameters with percentage of error [shown in bracket]				
		No. of measurements				
		6	5	4	3	2
$\theta_1 = 0.5$	GNA	0.515, 3.965 [3.013]	0.516, 5.882 [3.014]	0.481, 5.964 [3.965]	0.480, 6.141 [4.012]	0.479, 6.181 [4.209]
	IIRS	0.465, 8.189 [7.201]	0.461, 8.556 [8.204]	0.457, 8.663 [8.810]	0.552, 9.022 [10.403]	0.558, 9.166 [11.60]
$\theta_2 = 0.7$	GNA	0.722, 4.019 [3.143]	0.723, 5.930 [3.286]	0.725, 5.972 [3.571]	0.668, 6.152 [4.429]	0.669, 6.194 [4.430]
	IIRS	0.642, 7.854 [8.286]	0.640, 8.564 [8.713]	0.782, 8.795 [11.714]	0.785, 8.886 [12.413]	0.788, 9.104 [12.571]
$\theta_3 = 0.2$	GNA	0.206, 5.343 [3.011]	0.191, 5.362 [3.502]	0.191, 5.388 [3.502]	0.190, 7.319 [3.503]	0.190, 7.477 [3.504]
	IIRS	0.188, 6.881 [5.503]	0.188, 7.215 [6.018]	0.185, 7.373 [7.012]	0.219, 7.710 [8.505]	0.220, 8.101 [10.095]
$\theta_4 = 0.9$	GNA	0.931, 3.996 [3.444]	0.860, 4.013 [4.111]	0.859, 5.953 [4.444]	0.858, 6.129 [4.667]	0.857, 6.299 [4.778]
	IIRS	0.805, 8.847 [10.556]	0.799, 9.297 [11.09]	0.789, 9.510 [12.224]	0.775, 9.963 [13.667]	1.025, 10.220 [15.012]
$\theta_5 = 0.6$	GNA	0.619, 5.628 [3.167]	0.573, 5.787 [4.167]	0.574, 5.843 [4.167]	0.574, 5.968 [4.333]	0.574, 6.084 [4.333]
	IIRS	0.548, 6.862 [8.334]	0.543, 7.202 [9.013]	0.665, 7.485 [10.667]	0.667, 7.891 [11.014]	0.670, 8.331 [11.667]
$\theta_6 = 1.0$	GNA	0.956, 5.656 [4.207]	0.955, 5.647 [4.403]	0.954, 5.860 [4.508]	0.954, 5.919 [4.703]	0.952, 6.069 [4.808]
	IIRS	0.873, 9.275 [12.602]	0.868, 9.505 [13.013]	0.864, 9.991 [13.406]	0.845, 10.129 [15.312]	0.837, 10.424 [16.126]
$\theta_7 = 0.8$	GNA	0.825, 5.406 [3.014]	0.766, 5.673 [3.375]	0.767, 5.873 [4.125]	0.764, 6.067 [4.248]	0.827, 6.110 [4.505]
	IIRS	0.739, 6.876 [7.252]	0.881, 7.209 [9.875]	0.883, 7.491 [10.125]	0.884, 7.692 [10.375]	0.888, 7.997 [11.013]
$\theta_8 = 0.4$	GNA	0.382, 5.119 [2.752]	0.411, 5.303 [3.751]	0.385, 5.362 [3.762]	0.384, 5.474 [4.012]	0.382, 5.618 [4.018]
	IIRS	0.377, 6.805 [5.750]	0.374, 7.278 [5.752]	0.371, 7.740 [6.258]	0.436, 8.183 [9.011]	0.447, 8.744 [11.751]

7. Conclusions

In the present study, it can be observed that the model reduction technique using Gauss-Newton algorithm performs better over IIRS method and estimates natural frequencies from reduced order model quite accurately. It also estimates modal ordinates corresponding to unmeasured DOFs quite well. To test the effectiveness of the proposed model reduction algorithm, two cases are presented: (i) the number of modes are taken equal to the number of master DOFs and (ii) modes taken are less than the number of master DOFs being reduced gradually. The unknown modal ordinates for any mode number are determined based on the two cases as mentioned above. As expected, the unknown modal ordinates corresponding to slave DOFs are well estimated using number of modes equal as compared to the DOFs. However, it is also noted that the proposed model reduction algorithm estimates the unknown mode shape values for higher modes using reduced number of modes as compared to the DOFs quite accurately than the traditional IIRS based model reduction algorithm. The proposed model reduction algorithm is further tested by incorporating noises of varying percentages to the measured modal data for (i) and (ii). It is found that the estimated natural frequencies and modal ordinates corresponding to slave DOFs are measured quite well with percentages of error less than the percentage of noise added to the experimental data. It proves that the proposed model reduction algorithm is more robust compared to other model reduction algorithms. It is also noted that the Bayesian decision making statement is significantly enhanced with the unification of the modal ordinates for unmeasured DOFs in addition to the measured ones. Hence, the proposed model reduction coupled with MCMC algorithm gives a more accurate estimate of model parameters over MCMC run with full model using only measured DOF values in the Bayesian inference.

Disclosures

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