

Displacement in Deep Propped Cantilever Beam Subjected to Semi-elliptical Load

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Abstract

A fifth order shear deformation theory considering transverse shear deformation effect is presented for displacement in propped cantilever beam. The considered displacement field accounts for non-linear variation corresponding to depth of beam. Governing equations and boundary conditions of the theory are obtained using the standard of virtual work. Numerical results for static deflection analysis for aspect ratio (length to depth) 2 to 50 for isotropic beam is obtained. The results of present theory are in close agreement with those of higher order shear deformation theories.

Keywords: Deep beam, Fifth order shear deformation theory, semi-elliptical load, propped cantilever

1. Introduction

Beams are widely used in the engineering sectors. Theories of beams include the reduction of a 3D problem of elasticity theory to a 1D problem. Since the depth is much smaller than the length, it shows potential to estimate the variation of the displacement, strain, and stress components in the thickness dimension. Euler-Bernoulli [1, 2] (1705, 1744) developed the elementary theory of beam (ETB). It is based the assumption that the plane sections remain plane and normal to the axis after bending, implying that the transverse shear strain is zero. This theory neglects the transverse shear effect, it is suitable for the analysis of thin beams. The refined beam theory was initially developed by Timoshenko. This theory is known as first order shear deformation theory (FSDT). This theory gives constant shear strain throughout the thickness. A first order shear deformation beam theory is used by Ike [24] (2019) for the flexural analysis of moderately thick beams. In addition, ETB and FSDT have been unable to notice significant effects such as warping. Those effects are due to deep beam in nature. That has led to the development of higher order theories. Ghugal and Shimpi [6] (2002) have presented meticulous review of refined shear deformation theories.

Krishna Murty [7] (1984) and various researchers established the higher order shear deformation theories for beams, assuming a third order variation of axial displacement in terms of thickness coordinate. However, the issues of bending and transverse shear stress distributions at the built-in end boundary are not covered. Apart from these different theories are available in the literature by using trigonometric and hyperbolic functions in thickness

coordinate to demonstrate the warping effects. Ghugal and Sharma [8, 9] (2009, 2011) presented the hyperbolic shear deformation theory for the static and dynamic analysis of thick beams. Sawant and Dahake [18] (2016), Sawant *et al.* [19] (2016), Sawant and Nimbalkar [21] (2018), Saba and Mangulkar [22] (2018) employed the hyperbolic shear deformation theory to study the bending of deep cantilever isotropic beams subjected to parabolic load, varying load, cosine load, parabolic load respectively.

Unified shear deformation theory was used by Sayyad [10] (2011) to analyze the simply supported thick isotropic beams for transverse displacement, axial bending stress, transverse shear stress, and natural frequency. A third order shear deformation theory is developed by Dahake *et al.* [16] (2013) for flexure of thick beams, taking into account the effects of transverse shear deformation. Joshi *et al.* [26] (2020) presented the study of displacements in cantilever thick beam exposed to a cosine loading via 5th order function of study of shear deformation. It is also observed that the analytical solutions for static flexure of thick cantilever beams that are subjected to semi elliptical loading using 5th order shear deformation theories are very limited. Many researchers have reported their results for aspect ratio 4 and 10 only.

In this paper, the deep beam is subjected to semi elliptical loading to find deflection for propped cantilever beam is considered. Here, as an attempt is made to present the results for aspect ratio 2 to 50. The analytical result obtained by this theory and other refined theories are compared to validate the theory.

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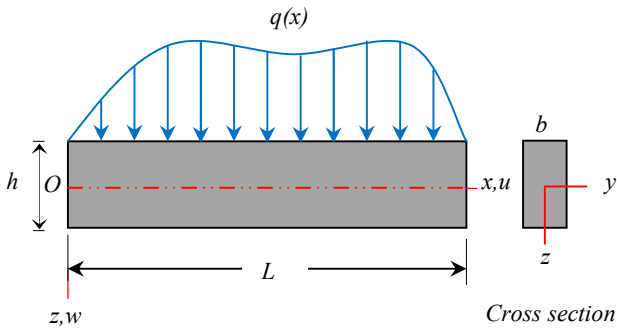


Fig. 1. Beam under bending in x-z plane

2. Analytical Formulation

The Beam Under Consideration

The beam under consideration as shown in Fig. 1

$x, y, z =$ Cartesian coordinates,

$L =$ Length in x direction

$B =$ Breadth in y directions and

$h =$ Thickness in the z -direction.

The beam is made up of homogeneous, linearly elastic isotropic material.

The analytical formulation of uniform thick propped cantilever beam based on certain kinematical and physical assumptions is presented.

Strain-Displacement Relationships

Normal and shear strains are obtained within the skeleton of linear theory of elasticity using the displacement field given by Eqn. (1). These relationships are given as follows

The Displacement Field

$$u(x, z) = u_0 - z \frac{dw}{dx} + z \left[2 - \frac{4}{3} \left(\frac{z}{h} \right)^2 - \frac{16}{5} \frac{z^4}{h^4} \right] \phi(x) \tag{1}$$

$$w(x, z) = w(x)$$

Normal Strain

$$\epsilon_x = \frac{du}{dx} = -z \frac{d^2w}{dx^2} + z \left[2 - \frac{4}{3} \left(\frac{z}{h} \right)^2 - \frac{16}{5} \frac{z^4}{h^4} \right] \frac{d\phi}{dx} \tag{2}$$

Shear Strain

$$\gamma_{zx} = \frac{du}{dz} + \frac{dw}{dx} = \left(2 - 4 \frac{z^2}{h^2} - 16 \frac{z^4}{h^4} \right) \phi(x) \tag{3}$$

Relationships between Stress-Strain

The Hooke's law is applied. The axial stress σ_x is related to strain ϵ_x and shear stress is related to shear strain by the following constitutive relations:

$$\sigma_x = E\epsilon_x = -zE \frac{d^2w}{dx^2} + zE \left[2 - \frac{4}{3} \frac{z^2}{h^2} - \frac{16}{5} \frac{z^4}{h^4} \right] \frac{d\phi}{dx} \tag{4}$$

$$\tau_{zx} = G\gamma_{zx} = G \left(2 - 4 \frac{z^2}{h^2} - 16 \frac{z^4}{h^4} \right) \phi(x) \tag{5}$$

where E is modulus of elasticity and G is shear modulus of the beam material.

Governing Equations and Boundary Conditions

Using the expressions for strains and stresses (2) through (5) and using the principle of virtual work, variationally consistent governing differential equations and boundary conditions for the beam under consideration can be obtained. The principle of virtual work when applied to the beam leads to:

$$b \int_{x=0}^{x=L} \int_{z=-h/2}^{z=+h/2} (\sigma_x \delta\epsilon_x + \tau_{zx} \delta\gamma_{zx}) dx dz = \int_{x=0}^{x=L} q(x) \delta w dx \tag{6}$$

Substituting expressions for the virtual strains and stresses [(2) through (5)] into Eqn. (6) we get following equation

$$b \int_{x=0}^{x=L} \int_{z=-h/2}^{z=+h/2} \left[\begin{aligned} & \left(-zE \frac{d^2w}{dx^2} + zE \left(2 - \frac{4}{3} \frac{z^2}{h^2} - \frac{16}{5} \frac{z^4}{h^4} \right) \frac{d\phi}{dx} \right) \\ & \left(-z \frac{d^2\delta w}{dx^2} + 2 - \frac{4}{3} \frac{z^2}{h^2} - \frac{16}{5} \frac{z^4}{h^4} \frac{d\delta\phi}{dx} \right) \\ & + G \left(2 - 4 \frac{z^2}{h^2} - 16 \frac{z^4}{h^4} \right) \phi \left(2 - 4 \frac{z^2}{h^2} - 16 \frac{z^4}{h^4} \right) \delta\phi \end{aligned} \right] dz dx = \int_{x=0}^{x=L} q(x) \delta w dx \tag{7}$$

Further simplifying it leads to

$$b \int_{x=0}^{x=L} \int_{z=-h/2}^{z=+h/2} \left[\begin{aligned} & z^2 E \frac{d^2w}{dx^2} \frac{d^2\delta w}{dx^2} \\ & -z^2 E \left(2 - \frac{4}{3} \frac{z^2}{h^2} - \frac{16}{5} \frac{z^4}{h^4} \right) \frac{d\phi}{dx} \frac{d^2\delta w}{dx^2} \\ & -z^2 E \left(2 - \frac{4}{3} \frac{z^2}{h^2} - \frac{16}{5} \frac{z^4}{h^4} \right) \frac{d^2w}{dx^2} \frac{d\delta\phi}{dx} \\ & +z^2 E \left(2 - 4 \frac{z^2}{h^2} - 16 \frac{z^4}{h^4} \right)^2 \frac{d\phi}{dx} \frac{d\delta\phi}{dx} \\ & + G\phi\delta\phi \left(2 - 4 \frac{z^2}{h^2} - 16 \frac{z^4}{h^4} \right)^2 \end{aligned} \right] dz dx = \int_{x=0}^{x=L} q(x) \delta w dx \tag{8}$$

Integration of Eqn. (8) with respect to z leads to following equation

$$\int_{x=0}^{x=L} \left[\begin{aligned} & EI \frac{d^2w}{dx^2} \frac{d^2\delta w}{dx^2} - \frac{12}{7} EI \frac{d\phi}{dx} \frac{d^2\delta w}{dx^2} \\ & - \frac{12}{7} EI \frac{d^2w}{dx^2} \frac{d\delta\phi}{dx} + 2.96EI \frac{d\phi}{dx} \frac{d\delta\phi}{dx} \\ & + 2.4635GA\phi\delta\phi \end{aligned} \right] dx = \int_{x=0}^{x=L} q(x) \delta w dx \tag{9}$$

and integrating the each term by parts in Eqn. (9) to relieve the virtual displacements (δw and $\delta \phi$) in the region of the beam (region of any differentiation), so that we can use the fundamental lemma of variational calculus, we obtain...

$$\int_{x=0}^{x=L} \left[EI \frac{d^4 w}{dx^4} - \frac{12}{7} EI \frac{d^3 \phi}{dx^3} - q(x) \right] \delta w dx + \int_{x=0}^{x=L} \left[\frac{12}{7} EI \frac{d^3 w}{dx^3} - 2.96 EI \frac{d^2 \phi}{dx^2} + 2.4635 GA \phi \right] \delta \phi dx + \left(EI \frac{d^2 w}{dx^2} - \frac{12}{7} EI \frac{d \phi}{dx} \right) \frac{d \delta w}{dx} \Big|_0^L + \left(-EI \frac{d^3 w}{dx^3} + \frac{12}{7} EI \frac{d^2 \phi}{dx^2} \right) \delta w \Big|_0^L + \left(-\frac{12}{7} EI \frac{d^2 w}{dx^2} + 2.96 EI \frac{d \phi}{dx} \right) \delta \phi \Big|_0^L = 0 \tag{10}$$

Since the variations (δw and $\delta \phi$) are arbitrary functions over the region of integration, we set the coefficients of their variations equal to zero separately to obtain the Euler-Lagrange equations (equilibrium equations or field equations) which are the governing differential equations and associated boundary conditions of the present theory of beam. The governing differential equations obtained are as follows:

$$EI \frac{d^4 w}{dx^4} - \frac{12}{7} EI \frac{d^3 \phi}{dx^3} - q(x) = 0 \tag{11}$$

$$\frac{12}{7} EI \frac{d^3 w}{dx^3} - 2.96 EI \frac{d^2 \phi}{dx^2} + 2.4635 GA \phi = 0 \tag{12}$$

The General Solution of Governing Equilibrium Equations of The Beam

The general solution for transverse displacement $w(x)$ and warping function $\phi(x)$ is as follows

$$\frac{d^3 w}{dx^3} = \frac{12}{7} \frac{d^2 \phi}{dx^2} + \frac{Q(x)}{EI} \tag{13}$$

where $Q(x)$ is the generalized shear force for beam and it is given by $Q(x) = \int^x q dx + C_1$.

Now second governing Eqn. (12) is rearranged in the following form:

$$\frac{12}{7} \frac{d^3 w}{dx^3} - 2.96 \frac{d^2 \phi}{dx^2} = -\beta \phi \tag{14}$$

Let, $A_0 = \frac{12}{7}$, $B_0 = 2.96$, $C_0 = 2.4635$

A single equation in terms of shear slope is now obtained as:

$$\frac{d^2 \phi}{dx^2} - \lambda^2 \phi = \frac{Q(x)}{\alpha EI} \tag{15}$$

where the constants α , β and λ appeared in are as follows

$$\alpha = \left(\frac{B_0}{A_0} - A_0 \right), \quad \beta = \frac{C_0 GA}{A_0 EI},$$

$$\text{and } \lambda^2 = \frac{\beta}{\alpha} \tag{16}$$

The general solution is given by:

$$\phi(x) = C_2 \cosh \lambda x + C_3 \sinh \lambda x - \frac{Q(x)}{\beta EI} \tag{17}$$

The general solution for transverse displacement $w(x)$ can be obtained as follows

$$w(x) = \frac{1}{D} \iiint q dx dx dx + \left[\frac{c_1 x^3}{6} + \left(\frac{A_0 \lambda^2 - \beta}{B_0} \right) \frac{EI}{\lambda^3} (c_2 \sinh \lambda x - c_3 \cosh \lambda x) + c_4 \frac{x^2}{2} + c_5 x + c_6 \right] \tag{18}$$

where C_1, C_2, C_3, C_4, C_5 and C_6 are the arbitrary constants of integration and can be obtained by imposing natural (forced) and /or geometric or kinematical boundary / end conditions of beam.

Illustrative Example

In order to validation of the present theory, an example is considered which is having the material properties as follows:

Young's modulus (E) = 2.1×10^5 MPa, Poisson's ratio (μ) = 0.3 and the density, $\rho = 7800$ kg/m³

The kinematic and static (forced) boundary conditions associated with propped cantilever beam (Example 1) bending problem is as follows:

At fixed support, $\frac{dw}{dx} = \phi = w = 0$ at $x = 0, L$

At propped support, $EI \frac{d^2 w}{dx^2} = EI \frac{d \phi}{dx} = w = 0$ at $x = L$

The analysis of beam bending problems with semi-elliptical loading and propped cantilever boundary conditions is presented as follows.

Example : A Beam With Propped Cantilever Subjected To Semi-Elliptical Load

A beam with propped cantilever has its origin at left hand side support. The beam is subjected to semi elliptical load on surface $z = +h/2$ acting in the downward z direction as shown in Fig. 2.

$$q(x) = q_0 \sqrt{1 - \left(\frac{2x}{L} \right)^2}$$

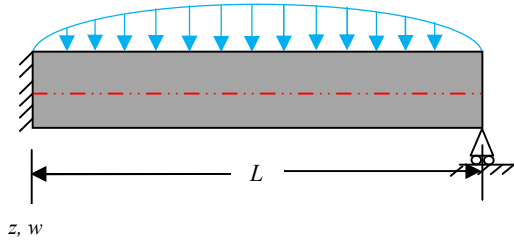


Fig. 2. A propped cantilever beam subjected to semi-elliptical load

General expressions using above boundary conditions obtained for $w(x)$ and $\phi(x)$ for given Example are as follows:

$$w(x) = \frac{q_0 L^4}{120EI} \left[\frac{1}{24} \frac{A_0^2 E h^2}{C_0 G L^2} \left(\frac{\cosh \lambda x - \sinh \lambda x - 1}{\lambda L} + \frac{x}{L} \right) - \frac{B_0 E h^2}{C_0 G L^2} \left[\frac{15x}{2L} \sin^{-1} \left(\frac{2x}{L} \right) + \frac{15\sqrt{L^2 - 4x^2}}{4L} - \frac{5(L^2 - 4x^2)^{3/2}}{6L^3} - \frac{35}{12} \right] + 5 \frac{x^3}{L^3} \sin^{-1} \left(\frac{2x}{L} \right) + \frac{5}{12} (L^2 + 2x^2) \sqrt{L^2 - 4x^2} \right] + \frac{15x}{16L} \sin^{-1} \left(\frac{2x}{L} \right) + \frac{15\sqrt{L^2 - 4x^2}}{32L} - \frac{15(L^2 - 4x^2)^{3/2}}{16L^3} + \frac{1(L^2 - 4x^2)^{5/2}}{4L^5} - \frac{3}{256} \cos \left[5 \sin^{-1} \left(\frac{2x}{L} \right) \right] - \frac{5}{256} \cos \left[3 \sin^{-1} \left(\frac{2x}{L} \right) \right] - \frac{1}{6} - \frac{1}{12} \frac{x^3}{L^3} + \frac{1}{4} \frac{x^2}{L^2} \right] \quad (19)$$

$$\phi(x) = \frac{q_0 L}{\beta EI} \left[\frac{1}{240} (\sinh \lambda x - \cosh \lambda x + 1) - \frac{1}{4} \sin^{-1} \left(\frac{2x}{L} \right) - \frac{1}{8} \sin \left(2 \sin^{-1} \left(\frac{2x}{L} \right) \right) \right] \quad (20)$$

3. Numerical Results

In this paper, the results for transverse displacement for aspect ratio 2 to 50 are presented for Example 1 in the following non-dimensional form for the purpose of presenting the results in this work. For beams subjected to load, $q(x)$

$$\bar{w} = \frac{10Ebh^3 w}{q_0 L^4}$$

Results obtained are presented in Table 1.

The variations of various displacements for propped cantilever beam is presented graphically in Fig. 3:

Table 1. Non-dimensional transverse deflection (\bar{w}) at ($x = 0.5L, z = 0.0$) of the beam with propped cantilever subjected to semi-elliptical load for aspect ratio 2 to 50.

Source	Bernoulli-Euler	Krishna Murty	Ghugal & Sharma	Present
Model	ETB	HSDT	HPSDT	V Order
Aspect Ratio	Non-Dimensional Displacement			
2	1.6035	3.9339	3.9257	3.9106
4	1.6035	2.1859	2.1839	2.1802
6	1.6035	1.8623	1.8614	1.8598
8	1.6035	1.7491	1.7486	1.7476
10	1.6035	1.6967	1.6963	1.6957
12	1.6035	1.6682	1.668	1.6675
14	1.6035	1.651	1.6509	1.6505
16	1.6035	1.6399	1.6397	1.6395
18	1.6035	1.6322	1.6321	1.632
20	1.6035	1.6268	1.6267	1.6265
22	1.6035	1.6227	1.6227	1.6225
24	1.6035	1.6197	1.6196	1.6195
26	1.6035	1.6173	1.6172	1.6171
28	1.6035	1.6154	1.6153	1.6152
30	1.6035	1.6138	1.6138	1.6137
32	1.6035	1.6126	1.6125	1.6125
34	1.6035	1.6115	1.6115	1.6115
36	1.6035	1.6107	1.6106	1.6106
38	1.6035	1.6099	1.6099	1.6099
40	1.6035	1.6093	1.6093	1.6092
42	1.6035	1.6088	1.6087	1.6087
44	1.6035	1.6083	1.6083	1.6082
46	1.6035	1.6079	1.6079	1.6078
48	1.6035	1.6075	1.6075	1.6075
50	1.6035	1.6072	1.6072	1.6072

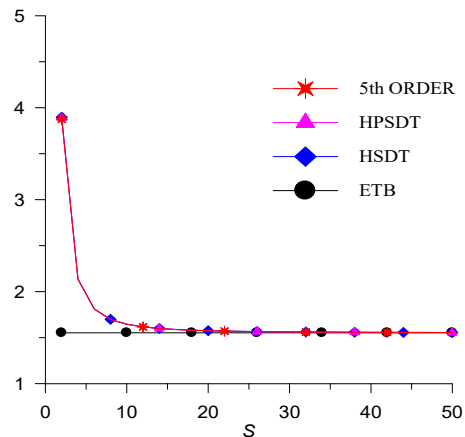


Fig. 3. Variation of longitudinal displacement with respect to the aspect ratio.

4. Discussion of Results

The results acquired from the present 5th order shear deformation theory (5th Order) in numerical and graphical forms are discussed with those of the elementary beam theory (ETB [1-2]), third order shear deformation theory (HSDT) of Krishna Murty (1984) and hyperbolic shear deformation theory (HPSDT) of Ghugal and Sharma (2009, 2011).

The comparison of results of maximum non-dimensional transverse displacement (\bar{W}) for the aspect ratios from 2 to 50 is presented in Table 1. Among the results of all the other theories, the values of present theory are in excellent agreement with the values of other refined theories for aspect ratios except those of classical beam theory (ETB).

5. Conclusions

- I. The transverse displacements obtained by the present theory are in excellent agreement with the other shear deformation theories.
- II. The distribution of transverse displacement with respect to aspect ratio is non-linear as well as rational.

The present theory provides precise results, as seen from the empirical example studied.

Disclosures

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