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Dynamic Analysis of Simply Supported Functionally Graded Plates

SourishDandapat[1,*], Dr Sreyashi Das[2], Subhankar Pramanik[3]

[1] Student ME Civil Engineering Jadavpur University
[2]Assistant Professor Jadavpur university department of Civil Engineering
[3]Student Ph.D. Civil Engineering Jadavpur University

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Abstract

Functionally graded materials (FGMs) are considered as a special composite material thatis synthesized for a specific purpose by gradual mixing of two or more different materials thus with varying material properties through their thickness. Due to smooth and continuous varying material properties from one face to another, FGMs are usually superior to the conventional composite materials in mechanical behavior. Hence the dynamic analysis of FGM plates is essential. In this paper, numerical solutions for free vibration analysis of moderately thick plates composed of functionally graded materialswith simply supported boundary conditions using finite element analysis have been studied. First-order shear deformation plate theory (FSDT) has been applied in the research.Eight noded isoparametric serendipity plate bending elements have been used.Young's modulus and density per unit volume are assumed to vary continuously through the plate thickness according to a power-law distribution.The results have been validated with the existing literature available from other analytical and numerical techniques. The effect of different plate parameters such as aspect ratios, edge constraints, thickness to length ratios, and gradient indices on the natural frequencies of FG rectangular plates are presented. It is found that the natural frequency depends considerably on the said parameters.

Keywords: Functionally graded, Free vibration, Finite element, First-order shear deformation

1. Introduction

Rapid development in science and technology has resulted in the development of various types of advanced materials. Composite materials are in this regard a special mention which has a wide variety of applications nowadays due to good strength to weight ratio. Functionally graded materials (FGM) is a class of two-component composite in which the mechanical property varies uniformly from one surface to the other. This concept has captured the interest of the scientific community, which has resulted in several investigations and technological applications such as in medical fields, defense sector, aerospace, energy motor, automobile engine, etc. In the aerospace industry, it is used for making the body of the airplane, components of a rocket motor, etc. In the medical field, FGM materials are used to replace tissues in bone and teeth by biomedical materials and serve as an alternate part. They are used in defense to make bulletproof vests.

Laminated composites that are extensively used nowadays in advanced engineering structures have high interlaminar stresses due to sudden change in material property across the interface which results in delamination. To overcome this problem FGM material is used currently and their dynamic analysis is relevant in recent research.

In 2000, Reddy [1] developed a finite element model based on third-order shear deformation theory to study the FGM plate structure. Vel et al. [2] studied free and forced vibration analysis of simply supported functionally graded plates using an exact three-dimensional solution. In 2005, Hashemi et al. [3] investigated the transverse vibration of a thick functionally graded plate using the Mindlin plate theory. Efraim and Eisenberger [4] studied free vibration of annular isotropic and FG plates with variable thickness and various boundary conditions using first-order shear deformation theory and exact element method to derive the stiffness matrix. In 2011, Hashemi et al., [5] studied the free vibration behavior of the FGM rectangular plate using FSDT. They had developed an exact closed-form free vibration equation of moderately thick Levy type FG rectangular plates using Mindlin's first-order shear deformation theory. Matsunaga [6] performed a dynamic and stability analysis of FG plates with a 2D higher-order shear deformation theory (HSDT). Talha et al. [7] studied static and free vibration response of FG plates with HSDT.

In this paper,free vibration analysis of moderately thick functionally graded plates with simply supported boundary conditions has been performed. Due to high efficiency and simplicity, first-order shear deformation plate theory has been used in the research.Eight noded isoparametric serendipity plate bending elements have been used.Young's modulus and density per unit volume are assumed to vary continuously through the plate thickness according to a power-law distribution.Parametric studies have been done for different boundary constraints, aspect ratio, thickness to length ratios, and power-law indices.

*Corresponding author. Tel: +918902675772; E-mail address:dandapat.sourish @gmail.com

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2 Theoretical Background

2.1 Geometrical Modelling

A typical FGM plate is shown in Fig. 1 with metal on one side and ceramic on the other. Hence the plate possesses the toughness of metal as well as refractoriness of ceramic. In other words, we can say metal in one layer gradually blends to ceramic in another layer. The Young's modulus, shear modulus, and the density per unit volume varied continuously from one surface to another using power-law distribution.

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^N$$

Similarly,

$$V_m = 1 - \left(\frac{z}{h} + \frac{1}{2}\right)^N \qquad (1)$$

where N is the chosen power-law index. The Material property at a specific z is given by,

$$P(z) = P_c V_c + P_m V_m \qquad (2)$$

Where z is the thickness direction, P_c, V_c, and P_m, V_m are the material properties and volume fractions of ceramic and metal respectively. Here the FGM material used is Al/Al_2O_3. the properties of the material are given in Table 1.

2.2 Displacement field

In the present finite element analysis first-order transverse shear deformation theory with constant transverse shear strain through the plate thickness has been used. The effect of rotary inertia is also taken in the element mass matrices. The displacement and rotation at any point is given by [8]

$$u(x,y,z) = u_0(x,y) + z\theta_y,$$
$$v(x,y,z) = v_0(x,y) - z\theta_x,$$
$$w = w_0$$
$$\emptyset_x = \theta_y + w_{,x},$$
$$\emptyset_y = -\theta_x + w_{,y} \qquad (3)$$

Table 1: Material properties of FGM materials

material	Modulus of elasticity (E) GPa	Density (ρ) Kg/m^3	Poisons ratio(μ)
Aluminum(Al)	70	2702	0.3
Alumina(Al_2O_3)	380	3800	0.3

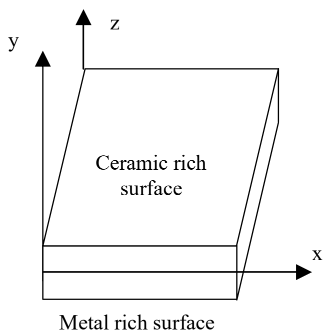

Fig. 1: Geometry of FGM plate

Here, $u_0(x,y), v_0(x,y)$ and w_0 are corresponding midplane displacements. θ_x and θ_y, are rotations alongthe x-axis and y-axis respectively. $\emptyset_x$ and $\emptyset_y$ are shear strain in x and y directions. linear strains at any point are expressed as [8]

$$\epsilon_x = u_{,x} = u_{0,x} + z\theta_{y,x} = \epsilon_x^0 + zK_x$$
$$\epsilon_y = v_{,y} = v_{0,y} - z\theta_{x,y} = \epsilon_y^0 + zK_y$$
$$\gamma_{xy} = u_{,y} + v_{,x} = u_{0,y} + v_{0,x} + z(\theta_{y,y} - \theta_{x,x})$$
$$= \epsilon_{xy}^0 + zK_{xy}$$
$$\gamma_{xz} = \emptyset_x \qquad (4)$$
$$\gamma_{yz} = \emptyset_y$$
$$\varepsilon_z = 0$$

where, K_x and K_y are the curvatures in x-z and y-z planes, respectively, whileK_{xy} is the cross curvature in the x-y plane.

The constitutive matrix [D] is given by [8,9]:

$$[D]=\begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{21} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{44} & A_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{45} & A_{55} \end{bmatrix} \qquad (5)$$

Here,$A_{ij}, B_{ij,}D_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} c_{ij}\,[1, z, z^2]$ dz (i, j = 1, 2 and 6)

and where $A_{ij} = \alpha \int_{-\frac{h}{2}}^{\frac{h}{2}} c_{ij}$dz ………(i, j = 4,5)

Here α is the shear correction factor taken as 5/6.

2.3 Finite Element Formulation

Using eight noded isoparametric elements, the linear stiffness matrix is given by

$[K_e]=\int_{-1}^{1}\int_{-1}^{1}[B]^T[D][B]|J|d\xi\, d\eta$(6)

where $[B_i]$ is a linear strain-displacement matrix given by [10], J is the Jacobian given by

$$[J]=\begin{bmatrix} \sum_1^{nel}\frac{\partial N_i}{\partial \xi}x_i & \sum_1^{nel}\frac{\partial N_i}{\partial \xi}y_i \\ \sum_1^{nel}\frac{\partial N_i}{\partial \eta}x_i & \sum_1^{nel}\frac{\partial N_i}{\partial \eta}y_i \end{bmatrix}$$

$$[B_i]=\begin{bmatrix} N_{i,x} & 0 & 0 & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 & 0 \\ N_{i,y} & N_{i,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{i,x} \\ 0 & 0 & 0 & -N_{i,y} & 0 \\ 0 & 0 & 0 & -N_{i,x} & N_{i,y} \\ 0 & 0 & N_{i,x} & 0 & N_i \\ 0 & 0 & N_{i,y} & -N_i & 0 \end{bmatrix} \quad (i = 1 \text{ to } 8)(7)$$

The element mass matrix with rotary inertia is given by [8,9], $[M_e] = \iint [N]^T [\rho][N]|J| d\xi\, d\eta$(8)

in which, the inertia matrix $[\rho] = \begin{bmatrix} I & & & & \\ 0 & I & Symm & & \\ 0 & 0 & I & & \\ P & 0 & 0 & Q & \\ 0 & P & 0 & 0 & Q \end{bmatrix}$,

where, $I, P, Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(z)\,(1, Z, Z^2) dz$(9)

$\rho(z)$ being the density at a distance z.N_i = Shape function at a node i. ξ, η = Local natural coordinates of the element.

The governing equation for free vibration analysis is given by

$([K'] - \omega_n{}^2[M])\{\delta\} = 0$(10)

3 Numerical Results and discussion

A MATLAB program with eight noded isoparametric plate bending elements is developed for this study. The study of validation with published literature is presented in this section. After verification of the proposed formulation, the program is used to study the dynamic behavior of the FGM moderately thick rectangular plate structure.

3.1 Mesh convergence

A simply supported FG (Al/Al_2O_3)square plate structure of side 1m is studied to check the mesh convergence. The number of elements along x and y axes is taken as 4, 6, 8, and 10 respectively. The first four natural frequencies are tabulated with a different number of mesh arrangements in Table 2. It is seen that a 10 x 10 discretization can be used for further analyses.

3.2 Validation problem

Square FG plate made up of $Al/Al_2O_3$with SSSS support conditions and having 1 m length along x and y-axis is analyzed and the natural frequency parameters are found out. Length to thickness ratio and N values are varied as follows:

a/h = 20, 10 and 5

N = 0, 0.5, 1, 4, 10 and ∞.

The non-dimensional first mode frequency parameter $\beta = \omega a^2 \sqrt{\frac{\rho_c}{E_c}}/h$ is shown and compared with Hashemi et al. [5] and Matsunaga et al. [6] in Table 3.

Table 2: First four non-dimensional natural frequencies ($\beta = \omega a^2 \sqrt{\frac{\rho_c}{E_c}}/h$) for square simply supported plate for different mesh arrangements

Mode number	Mesh size (4x4)	Mesh size (6x6)	Mesh Size (8x8)	Mesh Size (10x10)
1	2.840	2.836	2.835	2.835
2	4.616	4.533	4.525	4.523
3	8.019	7.405	7.339	7.323
4	9.672	9.552	9.532	9.527

Table 3: Comparison of the natural frequency parameter $\beta = \omega a^2 \sqrt{\frac{\rho_c}{E_c}}/h$ for SSSS condition of Al/Al_2O_3 of a square plate of side 1m

a/h	Method	N=0	N=0.5	N=1	N=4	N=10	N= ∞
20	Present	0.0148	0.0125	0.0113	0.0098	0.0094	------
	FSDT[5]	0.0148	0.0125	0.0113	0.0098	0.0094	
10	Present	0.0577	0.0490	0.0442	0.0382	0.0366	0.0294
	FSDT[5]	0.0577	0.0490	0.0442	0.0382	0.0366	0.0293
	HSDT[6]	0.0577	0.0492	0.0443	0.0381	0.0364	0.0293
5	Present	0.2112	0.1803	0.1628	0.1394	0.1323	0.1075
	FSDT[5]	0.2112	0.1805	0.1631	0.1397	0.1324	0.1076
	HSDT[6]	0.2121	0.1819	0.1640	0.1383	0.1306	0.1077

It is seen that the result matches very well with Hashemi et al. [5]. They used the first-order shear deformation theory in their study. Whereas solutions reported by Matsunaga [6] was based on higher-order deformation theory. The program developed here gives accurate solutions for low to moderate thickness.

3.3 Case Study 1: Study on square plate (a/b = 1)

In this part the free vibration characteristics of a square FG Al/Al_2O_3plate (a/b = 1) with each side 4m is found out for different length to thickness ratio and different power-law index N. Four different boundary conditions have been used in the analysis.

SSSS- all side simply supported

SCSC- Two opposite side simply supported, other two clamped

SCCC- One edge simply supported; three edges clamped

SSCC-Two adjacent edges simply supported, other two clamped

The first three non-dimensional frequency parameters are tabulated in Table 4. Fig. 2 shows the variation of the frequency parameter for different boundary conditions for a/h=10.

From Table 4, it is seen that with an increase in the value of N, stiffness reduces, and hence, the natural frequency decreases exponentially. With an increase in a/h ratio,

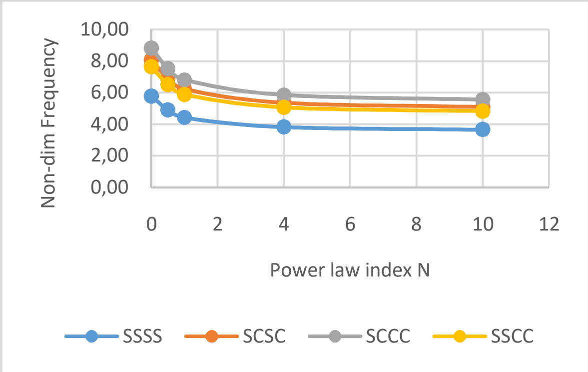

Fig. 2. Variation of non-dimensional frequency parameter for different boundary conditions for (*a/h* = 10) for a/b = 1

Table 4. First three non-dimensional frequency $\omega a^2\sqrt{\frac{\rho_c}{E_c}}/h$ for square Al/Al$_2$O$_3$ plate (*a/b* =1)

			N Values				
BC	a/h	mode	0	0.5	1	4	10
SSSS	5	1	5.28	4.51	4.07	3.48	3.31
		2	9.74	8.73	8.08	6.51	5.75
		3	9.74	8.73	8.08	6.51	5.75
	10	1	5.77	4.90	4.42	3.82	3.66
		2	13.77	11.73	10.58	9.10	8.67
		3	13.77	11.73	10.58	9.10	8.67
	20	1	5.92	5.02	4.52	3.93	3.77
		2	14.61	12.40	11.17	9.68	9.28
		3	14.61	12.40	11.17	9.68	9.28
SCSC	5	1	6.77	5.84	5.30	4.48	4.18
		2	9.74	8.73	8.08	6.51	5.75
		3	12.06	10.41	9.43	7.94	7.42
	10	1	8.07	6.88	6.22	5.35	5.08
		2	14.87	12.69	11.46	9.83	9.33
		3	17.93	15.37	13.91	11.87	11.19
	20	1	8.57	7.27	6.56	5.68	5.44
		2	16.07	13.64	12.30	10.65	10.19
		3	20.07	17.06	15.40	13.30	12.70
SCCC	5	1	7.33	6.33	5.74	4.85	4.52
		2	12.55	11.22	10.25	8.34	7.39
		3	13.06	11.34	10.41	8.64	8.00
	10	1	8.82	7.52	6.80	5.84	5.55
		2	16.75	14.33	12.96	11.08	10.48
		3	18.31	15.70	14.21	12.12	11.42
	20	1	9.40	7.98	7.20	6.24	5.97
		2	18.45	15.67	14.14	12.23	11.69
		3	20.54	17.47	15.76	13.62	13.00
SSCC	5	1	6.58	5.66	5.13	4.35	4.08
		2	11.10	9.94	9.19	7.40	6.54
		3	12.73	11.02	10.00	8.39	7.79
	10	1	7.65	6.52	5.88	5.07	4.83
		2	16.17	13.82	12.49	10.69	10.12
		3	16.27	13.90	12.57	10.76	10.19
	20	1	8.04	6.82	6.15	5.33	5.11
		2	17.69	15.03	13.55	11.73	11.21
		3	17.77	15.10	13.62	11.78	11.27

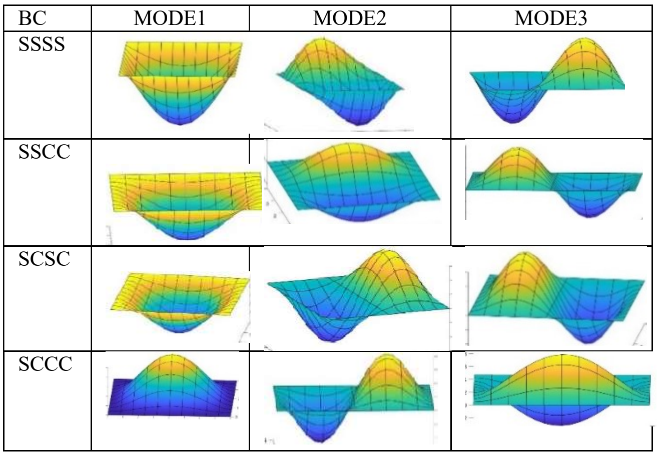

Fig. 3. First three mode shapes of square FG plate for *a/h*=20 for a/b=1

thickness decreases, hence natural frequency is reduced too. But the non-dimensional frequency parameter increases as the thickness is in the denominator. Natural frequency is least for SSSS conditions.

With an increase in boundary constraints, stiffness and hence natural frequency is increased. For SCCC, the stiffness is highest. Among SCSC and SSCC, the change in natural frequency is nominal. SCSC boundary condition imparts slightly higher stiffness than that of SSCC condition, though the degree of edge constraint is the same for both the cases. The first three mode shapes for different boundary conditions have been plotted in Fig. 3. From Fig. 3, it is seen that all mode shapes are bending mode. Second and third mode shapes are similar for all cases, though natural frequency is not the same for these modes except the SSSS case.

3.4 Case Study 2: Study on rectangular plate (a/b = 2)

In this part the free vibration characteristics of a square FG Al/Al$_2$O$_3$plate (*a/b* = 2, a=4m) is found out for different length to thickness ratio and different power-law index N. Four different boundary conditions have been used in the analysis.

SSSS- All side simply supported
SCSC-Two opposite short side simply supported
CSCS- Two opposite long side simply supported
SCCC- One short side simply supported
CSCC- One long side simply supported
SSCC- Two adjacent side simply supported

The first three non-dimensional frequency parameters are tabulated in Table 5. Fig. 5 shows the variation of the frequency parameter for different boundary conditions for a/h=10. It is observed that with an increase in N value, natural frequency reduces. Also, boundary conditions play a very important role in the stiffness of the plate. For the SCCC case, the stiffness is highest. It is quite obvious as in the SCCC case only the shorter side is simply supported. In terms of increase in stiffness the boundary cases can be written as SSSS <CSCS <SSCC <CSCC <SCSC <SCCC. From Fig. 4, it is seen that change in frequency is nominal for CSCC and SSCC, and, SCSC and SCCC. The first three mode shapes are plotted in Fig. 4. It is noticed that as the plate is rectangular with a/b=2, the first mode is a bending mode with a half-sine wave, the second mode is like the square plate with a full sine wave, and the third mode experiences three half-sine waves for all boundary conditions.

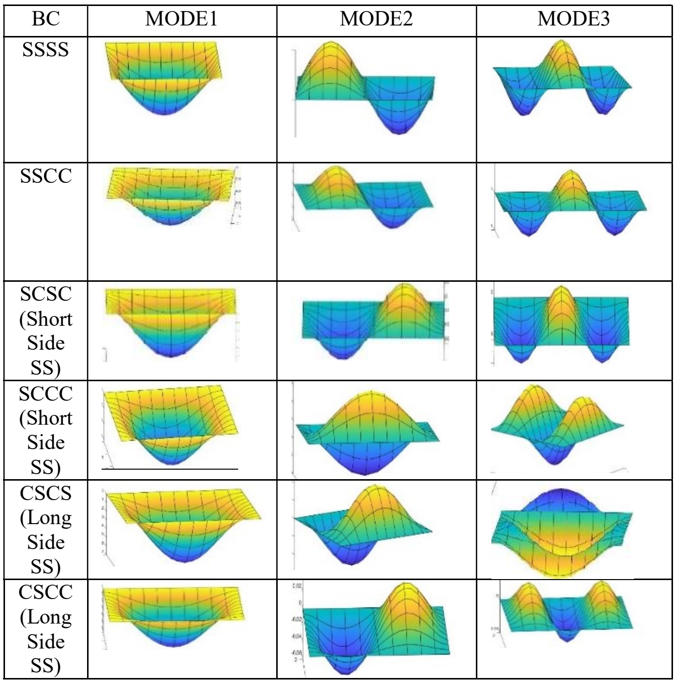

Fig. 4. First three mode shapes of square FG plate for a/h=20 for a/b=2

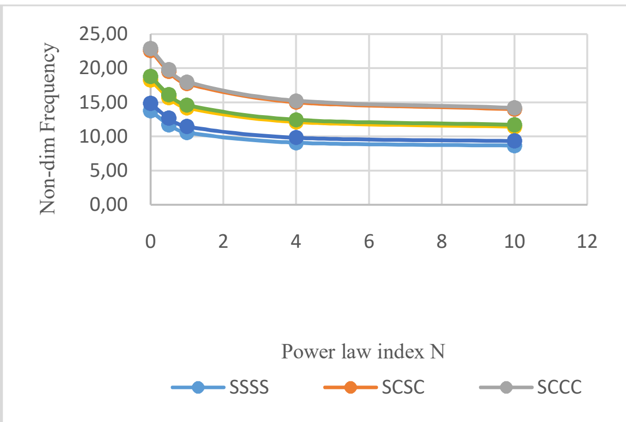

Fig. 5. Variation of non-dimensional frequency parameter for different boundary conditions for (a/h = 10) for a/b = 2

Table 5. First three non-dimensional frequency $\omega a^2 \sqrt{\frac{\rho_c}{E_c}}/h$ for square Al/Al_2O_3 plate (a/b =2)

BC	a/h	mode	N Values				
			0	0.5	1	4	10
SSSS	5	1	9.74	8.73	8.08	6.51	5.75
		2	11.55	9.93	8.98	7.59	7.13
		3	16.69	14.42	13.07	10.97	10.23
	10	1	13.76	11.72	10.58	9.10	8.67
		2	19.48	17.48	16.18	13.08	11.52
		3	21.13	18.04	16.30	13.94	13.23
	20	1	14.61	12.39	11.17	9.68	9.28
		2	23.08	19.60	17.67	15.29	14.63
		3	36.80	31.29	28.22	24.35	23.06
SCSC (Two opposite short side simply supported)	5	1	15.58	13.67	12.51	10.34	9.41
		2	18.84	16.47	15.01	12.44	11.38
		3	19.48	17.39	16.00	12.80	11.40
	10	1	22.60	19.51	17.72	15.00	13.98
		2	27.07	23.35	21.19	17.92	16.74
		3	35.86	30.91	28.01	23.67	22.14
	20	1	26.77	22.82	20.62	17.76	16.88
		2	32.29	27.54	24.88	21.41	20.34
		3	43.20	36.85	33.29	28.61	27.18
SCCC (One short side simply supported)	5	1	15.81	13.88	12.70	10.51	9.55
		2	19.43	17.01	15.52	12.84	11.74
		3	21.02	18.75	17.24	13.78	12.27
	10	1	22.90	19.77	17.95	15.20	14.17
		2	28.12	24.27	22.02	18.62	17.38
		3	37.59	32.45	29.43	24.83	23.16
	20	1	27.12	23.13	20.90	17.99	17.10
		2	33.65	28.71	25.94	22.31	21.19
		3	45.84	39.13	35.35	30.36	28.81
SSCC (Two adjacent edges simply supported)	5	1	13.73	11.95	10.87	9.07	8.36
		2	17.59	15.70	14.39	11.61	10.31
		3	18.26	15.93	14.57	12.05	11.09
	10	1	18.30	15.69	14.20	12.11	11.41
		2	25.02	21.49	19.45	16.54	15.55
		3	35.18	30.76	27.86	23.52	20.77
	20	1	20.53	17.45	15.75	13.61	12.99
		2	28.73	24.45	22.07	19.03	18.14
		3	42.48	36.20	32.68	28.12	26.75
CSCS (Two opposite long side simply supported)	5	1	9.74	8.73	8.08	6.51	5.75
		2	12.06	10.41	9.43	7.94	7.42
		3	17.72	15.40	13.99	11.67	10.79
	10	1	14.86	12.69	11.46	9.83	9.33
		2	19.48	17.48	16.19	13.08	11.52
		3	23.86	20.47	18.52	15.77	14.85
	20	1	16.07	13.64	12.30	10.65	10.19
		2	27.12	23.07	20.82	17.96	17.14
		3	38.97	34.97	32.41	26.18	23.06
CSCC (One long side simply supported)	5	1	14.01	12.20	11.11	9.25	8.52
		2	18.64	16.41	14.94	12.30	10.93
		3	18.82	16.67	15.36	12.43	11.40
	10	1	18.77	16.11	14.58	12.43	11.70
		2	26.32	22.64	20.51	17.41	16.33
		3	37.27	32.41	29.39	24.75	22.00
	20	1	21.15	17.98	16.23	14.02	13.38
		2	30.62	26.08	23.54	20.29	19.32
		3	45.59	38.90	35.14	30.19	28.67

4 Conclusion

In this paper, free vibration analysis of the rectangular FGM plate has been done using Mindlin's first-order shear deformation plate theory. Combinations of simply supported and clamped boundary conditions have been used in the study. A validation study is carried out to verify the accuracy of the present formulation. It is observed that with an increase in the value of power-law index N, natural frequency reduces exponentially. With an increase in thickness by side ratio, the stiffness of the structure and hence natural frequency increases. As a result, the non-dimensional frequency parameter decreases. An increase in the degree of boundary constraints also increases the stiffness of the structure. In the case of a rectangular plate with a/b=2, a nominal increase in stiffness is seen when the boundary of the short edge is changed from simply supported to clamped condition.

Disclosures

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